

## Solutions to Homework 5.

(1) Prove the *Five Lemma*:

Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \downarrow t_4 & & \downarrow t_5 \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

and prove:

- (a) [4pts] If  $t_2$  and  $t_4$  are surjective and  $t_5$  is injective, then  $t_3$  is surjective.
- (b) [4pts] If  $t_2$  and  $t_4$  are injective and  $t_1$  is surjective, then  $t_3$  is injective.
- (c) [2pts] If  $t_1, t_2, t_4$  and  $t_5$  are isomorphisms, then  $t_3$  is an isomorphism.

*Proof.* (a) Let  $b_3 \in B_3$ .

$$\begin{aligned}
 &\Rightarrow \exists a_4 \in A_4 \text{ with } t_4(a_4) = g_3(b_3) \text{ (} t_4 \text{ surjective)} \\
 &\Rightarrow g_4 g_3(b_3) = 0 = g_4 t_4(a_4) = t_5 f_4(a_4) \\
 &\Rightarrow f_4(a_4) = 0 \text{ (} t_5 \text{ injective)} \\
 &\Rightarrow \exists a_3 \in A_3 \text{ with } f_3(a_3) = a_4 \\
 &\Rightarrow g_3(b_3 - t_3(a_3)) = g_3(b_3) - g_3 t_3(a_3) = t_4(a_4) - t_4 f_3(a_3) = t_4(a_4) - t_4(a_4) = 0 \\
 &\Rightarrow \exists b_2 \in B_2 \text{ with } g_2(b_2) = b_3 - t_3(a_3) \\
 &\Rightarrow \exists a_2 \in A_2 \text{ with } t_2(a_2) = b_2 \text{ (} t_2 \text{ surjective)}
 \end{aligned}$$

Then

$$\begin{aligned}
 t_3(f_2(a_2) + a_3) &= t_3 f_2(a_2) + t_3(a_3) \\
 &= g_2 t_2(a_2) + t_3(a_3) \\
 &= g_2(b_2) + t_3(a_3) \\
 &= b_3 - t_3(a_3) + t_3(a_3) \\
 &= b_3
 \end{aligned}$$

(b) Let  $a_3 \in A_3$  with  $t_3(a_3) = 0$ .

$$\begin{aligned}
 &\Rightarrow t_4 f_3(a_3) = g_3 t_3(a_3) = 0 \\
 &\Rightarrow f_3(a_3) = 0 \text{ (} t_4 \text{ injective)} \\
 &\Rightarrow \exists a_2 \in A_2 \text{ with } f_2(a_2) = a_3 \\
 &\Rightarrow g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(a_3) = 0 \\
 &\Rightarrow \exists b_1 \in B_1 \text{ with } g_1(b_1) = t_2(a_2) \\
 &\Rightarrow \exists a_1 \in A_1 \text{ with } t_1(a_1) = b_1 \text{ (} t_1 \text{ surjective)} \\
 &\Rightarrow g_1 t_1(a_1) = g_1(b_1) = t_2 f_1(a_1) = t_2(a_2) \\
 &\Rightarrow f_1(a_1) = a_2 \text{ (} t_2 \text{ injective)} \\
 &\Rightarrow f_2(a_2) = f_2 f_1(a_1) = 0 = a_3
 \end{aligned}$$

Thus  $t_3$  is injective.

(c) trivial

(2)[10pts] Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $P$  and  $Q$  be projective  $R$ -modules. Show that  $Q \otimes_R P$  is a projective  $R$ -module.

*Proof.* Since  $P$  and  $Q$  are projective, there are projective  $R$ -modules  $P'$  and  $Q'$  such that  $P \oplus P' = F_1$  and  $Q \oplus Q' = F_2$  where  $F_1$  and  $F_2$  are free  $R$ -modules. Since the tensor product commutes with direct sums, the module  $F_1 \otimes_R F_2$  is free and

$$\begin{aligned} F_1 \otimes_R F_2 &\cong (P \oplus P') \otimes_R (Q \oplus Q') \\ &\cong (P \otimes_R (Q \oplus Q')) \oplus (P' \otimes_R (Q \oplus Q')) \\ &\cong (P \otimes_R Q) \oplus [(P \otimes_R Q') \oplus (P' \otimes_R (Q \oplus Q'))]. \end{aligned}$$

$P \otimes_R Q$  is projective.

(3)[10pts] Prove that every projective  $R$ -module has a free complement, i.e., there is a free  $R$ -module  $F$  so that  $P \oplus F$  is a free  $R$ -module.

*Proof.* Let  $Q$  be an  $R$ -module with  $P \oplus Q = F_1$ , where  $F_1$  a free  $R$ -module. Then:

$$\begin{aligned} P \oplus F_1^{\mathbb{N}} &= P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &= (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \\ &= F_1^{\mathbb{N}}. \end{aligned}$$

(4)[12pts] Let  $R$  be a Noetherian ring,  $M$  be a finite  $R$ -module, and  $\{N_i\}_{i \in I}$  a set of  $R$ -modules. Show:

$$\mathrm{Hom}_R(M, \oplus_{i \in I} N_i) \cong \oplus_{i \in I} \mathrm{Hom}_R(M, N_i).$$

*Proof.* Obviously,

$$\begin{aligned} \mathrm{Hom}_R(R, \oplus_{i \in I} N_i) &\cong \oplus_{i \in I} N_i \\ &\cong \oplus_{i \in I} \mathrm{Hom}_R(R, N_i). \end{aligned}$$

Thus for every finite free  $R$ -module  $F = R^r$ :

$$\mathrm{Hom}_R(F, \oplus_{i \in I} N_i) \cong \oplus_{i \in I} \mathrm{Hom}_R(F, N_i).$$

If  $M$  is a finite  $R$ -module, consider an exact sequence:

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where  $F_i$  are finite free  $R$ -modules. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M, \oplus_{i \in I} N_i) & \longrightarrow & \mathrm{Hom}_R(F_0, \oplus_{i \in I} N_i) & \longrightarrow & \mathrm{Hom}_R(F_1, \oplus_{i \in I} N_i) \\ \uparrow & & f \uparrow & & g \uparrow & & h \uparrow \\ 0 & \longrightarrow & \oplus_{i \in I} \mathrm{Hom}_R(M, N_i) & \longrightarrow & \oplus_{i \in I} \mathrm{Hom}_R(F_0, N_i) & \longrightarrow & \oplus_{i \in I} \mathrm{Hom}_R(F_1, N_i) \end{array}$$

Since  $g$  and  $h$  are isomorphisms, the five Lemma yields that  $f$  is an isomorphism.

(5)[10pts] Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Suppose that

$$0 \rightarrow K_1 \rightarrow P_1 \rightarrow M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K_2 \rightarrow P_2 \rightarrow M \rightarrow 0$$

are exact sequences with projective modules  $P_1$  and  $P_2$ . Show that  $K_1 \oplus P_2 \cong K_2 \oplus P_1$ .

*Proof.* Consider the diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{f_1} & P_1 & \xrightarrow{g_1} & M & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \text{id}_M \downarrow & & \\ 0 & \longrightarrow & K_2 & \xrightarrow{f_2} & P_2 & \xrightarrow{g_2} & M & \longrightarrow & 0 \end{array}$$

where the maps  $\alpha$  and  $\beta$  have to be constructed so that the diagram commutes. Since  $P_1$  is projective and  $g_2$  surjective there is a map  $\beta : P_1 \rightarrow P_2$  so that  $g_2\beta = g_1$  and the right square commutes. This implies that  $\text{im}(\beta f_1) \subseteq \ker(g_2) = \text{im}(f_2) \cong K_2$ . Thus there is a linear map  $\alpha : K_1 \rightarrow K_2$  so that  $\beta f_1 = f_2\alpha$  and the left square commutes. Consider the sequence:

$$(*) \quad 0 \rightarrow K_1 \xrightarrow{\theta} P_1 \oplus K_2 \xrightarrow{\tau} P_2 \rightarrow 0$$

where  $\theta$  and  $\tau$  are defined as follows:  $\theta(k_1) = (f_1(k_1), \alpha(k_1))$  and  $\tau(p_1, k_2) = \beta(p_1) - f_2(k_2)$ . Note that the sequence  $(*)$  is exact. Since  $P_2$  is projective, the assertion follows.

(6) Let  $R$  be a commutative domain,  $K$  its field of quotients. Prove:

- (a) [10pts] A torsion-free  $R$ -module  $M$  is injective if and only if  $M$  is divisible.  
 (b) [3pts]  $K$  is the injective hull of  $A$ .

*Proof.* (a) By (6.65) every injective module is divisible. It remains to show that a torsion-free divisible  $R$ -module  $M$  is injective. Let  $I \subseteq R$  be an ideal and  $f : I \rightarrow M$  an  $R$ -linear map. We need to show that  $f$  extends to a  $R$ -linear map  $g : R \rightarrow M$ . If  $f = 0$  there is nothing to show. If  $f \neq 0$  let  $a \in I$  with  $f(a) = m \neq 0$ . Since  $R$  is a domain, there is an element  $n \in M$  so that  $m = an$ . We claim that the map  $g : R \rightarrow M$  defined by  $g(1) = n$  extends  $f$ . Let  $b \in I$ , then  $f(ab) = af(b) = bm = abn = g(ab) = ag(b)$ . Since  $M$  is torsion-free,  $f(b) = g(b)$ . By (6.61)  $M$  is injective.

(b) By (a)  $K$  is an injective  $R$ -module. Since  $K$  is an essential extension of  $R$ , the quotient field  $K$  is the injective hull of  $R$ .

(7)[10pts] Let  $R$  be a Noetherian ring. Show that a direct sum of injective  $R$ -modules is an injective  $R$ -module.

*Proof.* Let  $\{E_i\}_{i \in J}$  be a set of injective  $R$ -modules and  $E = \bigoplus_{i \in J} E_i$ . By (6.61) we have to show that every  $R$ -linear map  $g : I \rightarrow E$  extends to an  $R$ -linear map  $f : R \rightarrow E$  for every ideal  $I$  of  $R$ . Let  $I \subseteq R$  be an ideal and  $g : I \rightarrow E$  an  $R$ -linear map. Since  $R$  is Noetherian,  $I$  is finitely generated, say:  $I = (a_1, \dots, a_m)$ . For all  $1 \leq k \leq m$  the  $i$ th component  $g(a_k)_i$  is zero for all but finitely many  $i \in J$ .

Let  $i_1, \dots, i_r \in J$  so that  $g(a_k)_i = 0$  for all  $i \in J - \{i_1, \dots, i_r\}$  and all  $1 \leq k \leq m$ . Then

$$\text{im}(g) \subseteq E' = \bigoplus_{j=1}^r E_{i_j} \subseteq E.$$

Let  $g' : I \rightarrow E'$  be the map defined by  $g'(a) = g(a)$  for all  $a \in I$ . Since the direct product of injective modules is injective,  $E'$  is an injective  $R$ -module and  $g'$  extends to an  $R$ -linear map  $f' : R \rightarrow E'$ .  $f'$  combined with the embedding  $\epsilon : E' \rightarrow E$  extends  $g$ .

(8)[10pts] Let  $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a local homomorphism of local Noetherian rings, and let  $N$  be an  $R$ -flat  $S$ -module such that  $N/\mathfrak{m}N$  has finite length (as an  $S$ -module). Show that for every finite length  $R$ -module  $M$ :

$$l_S(M \otimes_R N) = l_R(M)l_S(N/\mathfrak{m}N).$$

*Proof.* We show by induction on  $n = l_R(M)$  that  $l_S(M \otimes_R N) < \infty$  and that  $l_S(M \otimes_R N) = l_R(M)l_S(N/\mathfrak{m}N)$ . If  $n = 1$ , then  $M \cong R/\mathfrak{m}$  and  $M \otimes_R N \cong R/\mathfrak{m} \otimes_R N \cong N/\mathfrak{m}N$ .

Suppose that  $l_R(M) = n + 1$ . Since  $M$  is an  $R$ -module of finite length, we know that  $\mathfrak{m} \in \text{Ass}(M)$ . Thus there is an element  $v \in M$  so that  $U = Av \cong A/\mathfrak{m}$ . Consider the exact sequence  $0 \rightarrow U \rightarrow M \rightarrow M/U \rightarrow 0$ . Then  $l_R(U) = 1$  and  $l_R(M/U) = n$ . Since  $N$  is  $R$ -flat, the sequence

$$0 \rightarrow U \otimes_R N \rightarrow M \otimes_R N \rightarrow M/U \otimes_R N \rightarrow 0$$

is exact. By induction hypothesis  $l_S(U \otimes_R N) < \infty$  and  $l_S((M/U) \otimes_R N) < \infty$ , thus  $l_S(M \otimes_R N) < \infty$ . Moreover,

$$\begin{aligned} l_S(M \otimes_R N) &= l_S(U \otimes_R N) + l_S((M/U) \otimes_R N) \\ &= l_S(N/\mathfrak{m}N) + (l_R(M) - 1)l_S(N/\mathfrak{m}N) \\ &= l_R(M)l_S(N/\mathfrak{m}N). \end{aligned}$$

(9)[15pts] Let  $A$  be a commutative local ring,

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow 0$$

an exact sequence of finitely generated free  $A$ -modules. Prove that

$$\sum_{i=0}^n (-1)^i \text{rk}(F_i) = 0$$

where for a finitely generated free  $A$ -module  $F$   $\text{rk}(F)$  denotes the number of elements in a basis of  $F$ .

*Proof.* We use the following facts:

(i) If  $k$  is a field and

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

an exact sequence of finite dimensional  $k$ -vector spaces, then

$$\dim(V) = \dim(V') + \dim(V'').$$

(ii) Let  $A$  be a commutative ring and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a split exact sequence of  $A$ -modules. Then for every  $A$ -module  $N$  the sequence

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is split exact.

Since the rank of a free modules remains the same under localization, we may assume that  $(A, \mathfrak{m}, k)$  is a commutative local ring. Let

$$0 \rightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \rightarrow 0$$

be an exact sequence of finitely generated free  $A$ -modules. We proceed by induction on  $n$ . If  $n = 1$  the statement is trivial.

If  $n = 2$  the short exact sequence:

$$0 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

is split exact. Thus, after tensoring with  $\otimes_A k$  the sequence

$$0 \rightarrow F_2/\mathfrak{m}F_2 \rightarrow F_1/\mathfrak{m}F_1 \rightarrow F_0/\mathfrak{m}F_0 \rightarrow 0$$

is exact. By (i)

$$\dim(F_1/\mathfrak{m}F_1) = \text{rk}(F_1) = \dim(F_2/\mathfrak{m}F_2) + \dim(F_0/\mathfrak{m}F_0) = \text{rk}(F_2) + \text{rk}(F_0).$$

For the induction step, set  $P = \text{im}(f_2) = \ker(f_1)$  and consider the induced exact sequences:

$$0 \rightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_3} F_2 \rightarrow P \rightarrow 0$$

and

$$0 \rightarrow P \rightarrow F_1 \xrightarrow{f_1} F_0 \rightarrow 0.$$

Since  $F_0$  is a free  $A$ -module, the last sequence is split exact and  $P$  is a finitely generated projective  $A$ -module. By (6.58)  $P$  is a free  $A$ -module. By induction hypothesis

$$\text{rk}(P) + \sum_{i=1}^{n-1} (-1)^i \text{rk}(F_{i+1}) = 0$$

and

$$\text{rk}(P) = \text{rk}(F_1) - \text{rk}(F_0).$$

This yields the formula:

$$\sum_{i=0}^n (-1)^i \text{rk}(F_i) = 0.$$