Solutions to Homework 4.

(1) [10pts] Let R be a semilocal Noetherian ring and $I \subseteq R$ an ideal of R. Show that the following conditions are equivalent:

(a) I is an ideal of definition of R.
(b) $I \subseteq \mathrm{Jrad}(R)$ and R/I is an Artinian ring.
(c) $I \subseteq \mathrm{Jrad}(R)$ and R/I has finite length.
(d) $\mathrm{Supp}_R(R/I) = \mathrm{mSpec}(R)$.

Proof. (a) ⇒ (b) If I is an ideal of definition of R, then $\mathrm{rad}(I) = m_1 \cap \ldots m_r$ where $\mathrm{mSpec}(R) = \{m_1, \ldots, m_r\}$. R/I is a Noetherian ring with $\dim(R/I) = 0$. Thus R/I is Artinian.

(b) ⇔ (c) Theorem (5.53).

(c) ⇒ (d) Since $\dim(R/I) = 0$, $\mathrm{Supp}_R(R/I) \subseteq \mathrm{mSpec}(R)$. By assumption $I \subseteq \mathrm{Jrad}(R)$, hence $\mathrm{mSpec}(R) \subseteq V(I) = \mathrm{Supp}_R(R/I)$.

(d) ⇒ (a)

$$\mathrm{rad}(I) = \bigcap_{P \in V(I)} P = \bigcap_{m \in \mathrm{mSpec}(R)} = \mathrm{Jrad}(R).$$

Therefore I is an ideal of definition of R.

(2) [12pts] Let R be a discrete valuation domain. Show that the polynomial ring $R[x]$ has maximal ideals of height one and of height two.

Proof. Let $M \in \mathrm{mSpec}(R[x]$.

Case 1: $M \cap R = P \neq (0)$.

Since R is a DVR, $P = (p)$ is the maximal ideal of R. Then $Q = PR[x]$ is a prime ideal of $R[x]$ with $R[x]/Q \equiv K[x]$ where $K = R/P$, the residue class field of R. M corresponds to the maximal ideal M/Q of $K[x]$. Thus $M/Q = (f)$ is principal and M is generated by elements p and F, where $F \in R[x]$ is a polynomial with $f = F + Q$. By the generalized principle ideal theorem $\mathrm{ht}(M) \leq 2$. On the other hand, since $M \neq Q$, $\mathrm{ht}(M) \geq 2$.

Example: $M = (p, x)$

Case 2: $M \cap R = (0)$

In this case M generates a maximal ideal in $S^{-1}(R[x]) \equiv (S^{-1}R)[x]$ where $L = S^{-1}R$ is the field of quotients of R. Since $\dim(L[x]) = 1$, $\mathrm{ht}(M) = 1$.

Example: $M = (px - 1)$ where $P = (p)$ is the maximal ideal of R.

In order to see that $M = (px-1)$ is a maximal ideal of $R[x]$ consider the injective natural morphism:

$$\varphi : R \longrightarrow R[x]/(px - 1).$$

Since $px + (px - 1) = 1 + (px - 1)$ in $R[x]/(px - 1)$, the element $\varphi(p) = p + (px - 1)$ is invertible in $R[x]/(px - 1)$ and φ factors through R_p:

$$\tilde{\varphi} : R_p \longrightarrow R[x]/(px - 1).$$

Since $\tilde{\varphi}(1/p) = x + (px - 1)$, the morphism $\tilde{\varphi}$ is an isomorphism. Then M is a maximal ideal of $R[x]$ since $R_p = L$ is the field of quotients of R.

(3) [8pts] Let $R \subseteq S$ be an extension of rings such that the set $S - R$ is closed under multiplication. Show that R is integrally closed in S.

Proof. Let $b \in S - (0)$ be integral over R. Then there is a minimal integer $n \in \mathbb{N}$ so that b satisfies an integral equation of degree n:

$$b^n + a_{n-1}b^{n-1} + \ldots + a_1 b + a_0 = 0 \quad \text{with} \quad a_i \in R.$$

Since n is minimal, $b \in R$ if and only if $n = 1$. If $b \notin R$ and $n > 1$, then

$$b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_1 \notin R,$$

but

$$b(b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_1) = -a_0 \in R,$$

a contradiction. Hence $n = 1$ and $b \in R$.

(4) [10pts] Let R be a normal domain, $K = Q(R)$ its field of quotients, and $f(x) \in R[x]$ a monic polynomial. Show that $f(x)$ is irreducible in $K[x]$ if and only if $f(x)$ is irreducible in $R[x]$.

Proof. Let $\bar{K}$ be the algebraic closure of K and suppose that $f = gh$ with $g, h \in K[x]$ monic polynomials and g irreducible in $K[x]$. If $\alpha \in \bar{K}$ is a root of g, then $f(\alpha) = 0$ and α is integral over R. Since R is normal, the minimal polynomial of α has coefficients in R. Thus $g \in R[x]$ and hence $h \in R[x]$ by the division algorithm (since g is monic!) and f is reducible in $R[x]$.

(5) Let $R \subseteq S$ be an extension of rings with S integral over R. Show:

(a) [4pts] If $a \in R$ is a unit in S then a is a unit in R.

(b) [8pts] The Jacobson radical of R is the contraction of the Jacobson radical of S.

Proof. (a) If $a \in R$ is not a unit (in R) then $a \in P$ for some prime ideal $P \subseteq R$. Since $R \subseteq S$ is an integral extension there is a prime ideal $Q \subseteq S$ with $Q \cap R = P$. Hence a is not a unit in S.

(b) For every maximal ideal $m \subseteq R$ there is a prime ideal $Q \subseteq S$ with $Q \cap R = m$. Since $R \subseteq S$ is integral, Q is a maximal ideal of S. Moreover, if $M \subseteq S$ is a maximal ideal of S then $m = M \cap R$ is a maximal ideal of R. Thus:

$$\begin{aligned}
\text{Jrad}(S) \cap R &= \Big(\bigcap_{M \in \text{mSpec}(S)} M\Big) \cap R \\
&= \bigcap_{M \in \text{mSpec}(S)} (M \cap R) \\
&= \bigcap_{m \in \text{mSpec}(R)} m \\
&= \text{Jrad}(R)
\end{aligned}$$

(6) [10pts] Let $R \subseteq S$ be an extension of rings with S finitely generated and integral over R. Show that for every prime ideal $P \subseteq R$ there are only finitely many prime ideals $Q \subseteq S$ which lie over P.

Proof. Let $P \subseteq R$ be a prime ideal and let $W = R - P$ be the multiplicative set given by P. The extension

$$R/P \hookrightarrow S/PS$$

is integral and finitely generated and so is the localized extension:

$$k(P) = W^{-1}(R/P) \hookrightarrow W^{-1}(S/PS) = T.$$

Since $k(P)$ is a field, T is a ring of dimension 0. Moreover, T is finitely generated over $k(P)$, thus T is a Noetherian ring of dimension 0. Hence T is Artinian with only finitely many prime ideals. The prime ideals of T correspond to the prime ideals of S which lie over P.

(7) [20pts] Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial and let $Y = Z(f)$ be the algebraic variety defined by f. Y is called *non-singular* or *smooth* at a point $P \in Y$ if not all of the partial derivatives $\partial f/\partial x_i$ vanish at P. Let $A(Y)$ be the coordinate ring of Y and let $\mathfrak{m}_P \subseteq A(Y)$ be the maximal ideal of $A(Y)$ corresponding to P (that is, if $P = (a_1, \ldots, a_n)$, then $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)/(f)$). Show that Y is smooth at P if and only if the ring $A(Y)_{\mathfrak{m}_P}$ is regular.

Proof. In the following set $R = \mathbb{C}[x_1, \ldots, x_n]$. First note that there are the following equivalences:

$$\begin{aligned} P = (a_1, \ldots, a_n) \in Y = Z(f) &\Leftrightarrow f(a_1, \ldots, a_n) = 0 \\ &\Leftrightarrow f(x_1, \ldots, x_n) \in (x_1 - a_n, \ldots, x_n - a_n) \\ &\Leftrightarrow f(x_1, \ldots, x_n) = \sum_{i=1}^{n} h_i(x_i - a_i) \end{aligned}$$

where $h_i \in R$. (Note that the forward direction is an application of Taylor's formula.) Thus for all $1 \leq i \leq n$:

$$\partial f/\partial x_i = \sum_{j=1}^{n} \partial h_j/\partial x_i(x_j - a_j) + h_i.$$

Thus P is a non-singular point of Y if and only if $h_i(a_1, \ldots, a_n) \neq 0$ for some $1 \leq i \leq n$, or equivalently, $h_i \notin (x_1 - a_1, \ldots, x_n - a_n)$ for some $1 \leq i \leq n$. Set $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n) \subseteq R$.
Claim: $h_i \notin \mathfrak{m} \Leftrightarrow$ the maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is generated by $x_1 - a_1, \ldots, x_{i-1} - a_{i-1}, f, x_{i+1} - a_{i+1}, \ldots, x_n - a_n$.
Proof of Claim: " $\Rightarrow$ " Since h_i is a unit in $R_{\mathfrak{m}}$:

$$\begin{aligned} &(x_1 - a_1, \ldots, x_{i-1} - a_{i-1}, f, x_{i+1} - a_{i+1}, \ldots, x_n - a_n)R_{\mathfrak{m}} = \\ &(x_1 - a_1, \ldots, x_{i-1} - a_{i-1}, h_i(x_i - a_i), x_{i+1} - a_{i+1}, \ldots, x_n - a_n) = \\ &(x_1 - a_1, \ldots, x_{i-1} - a_{i-1}, x_i - a_i, x_{i+1} - a_{i+1}, \ldots, x_n - a_n) = \\ &\mathfrak{m}R_{\mathfrak{m}} \end{aligned}$$

" $\Leftarrow$ " If $h_i \in \mathfrak{m}$ for all $1 \leq i \leq n$, then $f \in \mathfrak{m}^2$ and $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m}/((f) + \mathfrak{m}^2)$. Thus the maximal ideal of $R_{\mathfrak{m}}/(f)R_{\mathfrak{m}}$ is minimally generated by n elements ($\dim(\mathfrak{m}/(f) + \mathfrak{m}^2) = n$) and f is not part of a minimal system of generators of $\mathfrak{m}$. This shows the claim.

Thus P is a smooth point of Y if and only if f is part of a minimal system of generators of $\mathfrak{m}R_{\mathfrak{m}}$ or, equivalently, if and only if $\operatorname{edim}((R/(f))_{\mathfrak{m}}) = n - 1 =$

$\dim((R/(f))_{\mathfrak{m}})$. Since $R/(f) = A(Y)$ we have that P is smooth on Y if and only if $A(Y)$ is a regular local ring at P.

(8) [18pts] Let $K \subseteq L$ be an extension of fields, $Q \subseteq L[x_1, \ldots, x_n]$ a prime ideal in the polynomial ring in n variables over L, and $P = Q \cap K[x_1, \ldots, x_n]$ its contraction to the polynomial ring over K. Show that $\mathrm{ht}Q \geq \mathrm{ht}P$ and that equality holds if L is algebraic over K. Use this to show that if two polynomials $f, g \in K[x_1, \ldots, x_n]$ have no common divisor in $K[x_1, \ldots, x_n]$ then f and g have no common divisor in $L[x_1, \ldots, x_n]$.

Proof. Consider the extension of rings:

$$A = K[x_1, \ldots, x_n]/P \subseteq B = L[x_1, \ldots, x_n]/Q.$$

Suppose (after renumbering if necessary) that $x_1 + P, \ldots, x_r + P$ is a transcendence basis of A over K. Thus for $r+1 \leq i \leq n$ the element $x_i + P$ is algebraic over $K(x_1 + P, \ldots, x_r + P) \subseteq Q(A)$, where $Q(A)$ is the quotient field of A. This implies that $x_i + Q \in B$ is algebraic over $L(x_1 + Q, \ldots, x_r + Q)$ for all $r + 1 \leq i \leq n$. Therefore

$$\dim(B) = \mathrm{trdeg}_L(B) \leq \mathrm{trdeg}(A) = \dim(A).$$

By problem (9) 0f Homework #3

$$n - \mathrm{ht}Q = \dim(B) \leq n - \mathrm{ht}P = \dim(A)$$

and thus $\mathrm{ht}Q \geq \mathrm{ht}P$. If $K \subseteq L$ is algebraic, the extension $K[x_1, \ldots, x_n] \subseteq L[x_1, \ldots, x_n]$ is integral. By (5.25): $\mathrm{ht}P \geq \mathrm{ht}Q$ and hence $\mathrm{ht}Q = \mathrm{ht}P$.

Suppose that $K \subseteq L$ is algebraic and that $q \in L[x_1, \ldots, x_n]$ is a prime element with $q \,|\, f$ and $q \,|\, g$. The prime element q generates the height one prime ideal $Q = (q) \in L[x_1, \ldots, x_n]$. Thus $P = Q \cap K[x_1, \ldots, x_n]$ is a prime ideal of height one which is principal, $P = (p)$. Since $f, g \in P$, f and g have the common divisor p in $K[x_1, \ldots, x_n]$.