Solutions to Homework 3.

(1) [6pts] For a polynomial $P(t) \in \mathbb{Q}$ show that the following conditions are equivalent:

(a) $P(n) \in \mathbb{Z}$ for all integers $n \in \mathbb{Z}$.
(b) $P(n) \in \mathbb{Z}$ for all but finitely many integers $n \in \mathbb{Z}$.
(c) $P(t) = \sum_{i=0}^{n} a_i \binom{t}{i}$ with $a_i \in \mathbb{Z}$ and $n \in \mathbb{N}$ suitable.

Proof. (a) $\Leftarrow$ (b) trivial
(b) $\Leftarrow$ (c) Note that the set $\{\binom{t}{i}\}_{i\in\mathbb{N}_0}$ is a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}[t]$, where $\binom{t}{0} = 1$ and $\binom{t}{i} = (1/i!)t(t-1)\dots(t-i+1)$ for $i > 0$. Write $P(t) = \sum_{i=0}^{n} a_i \binom{t}{i}$ where $a_i \in \mathbb{Q}$ and $a_n \neq 0$. We proceed by induction on $n = \deg(P(t))$. For the induction step consider the polynomial $Q(t) = P(t+1) - P(t)$. Then

$$Q(t) = \sum_{i=0}^{n} a_i[\binom{t+1}{i} - \binom{t}{i}] = \sum_{i=1}^{n} a_i \binom{t}{i-1}.$$

Thus $\deg(Q(t)) = n-1$ and by induction hypothesis $a_1, \dots, a_n \in \mathbb{Z}$. This implies that $a_0 \in \mathbb{Z}$.
(c) $\Leftarrow$ (a) trivial

(2) [12pts] Show:

(a) A Noetherian topological space is quasi-compact, that is, every open cover has a finite subcover.
(b) Any subset of a Noetherian topological space is Noetherian.
(c) A Hausdorff Noetherian space is a finite set with the discrete topology.

Proof. (a) Let X be a Noetherian topological space and $X = \cup_{i\in I} U_i$ an open cover of X. Construct an ascending chain of open subsets as follows: If $i_1 \in I$ with $U_{i_1} \neq X$ then there is an $i_2 \in I$ so that $U_{i_2} \not\subseteq U_{i_1}$. Then $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$. Suppose $i_1, \dots, i_m$ have been chosen so that for all $1 < k \leq m$

$$U_{i_1} \cup \ldots \cup U_{i_{k-1}} \subsetneq U_{i_1} \cup \ldots U_{i_k}.$$

If $X = U_{i_1} \cup \ldots \cup U_{i_m}$ we are done. Otherwise there is an $i_{m+1} \in I$ so that $U_{i_{m+1}} \not\subseteq U_{i_1} \cup \ldots \cup U_{i_m}$, etc. Since every ascending chain of open sets in X is stationary, this process stops after finitely many steps with $X = U_{i_1} \cup \ldots \cup U_{i_\ell}$.
(b) Let X be a Noetherian topological space and $Y \subsetneq X$ a nonempty subset. Suppose that for all $i \in \mathbb{N}$ there are given open subsets $\tilde{U}_i \subseteq Y$ of Y so that

$$\tilde{U}_1 \subseteq \tilde{U}_2 \subseteq \ldots \subseteq \tilde{U}_n \subseteq \ldots$$

is an increasing chain of open subsets of Y. Then there are open subsets $U_i \subseteq X$ of X so that $\tilde{U}_i = U_i \cap Y$. Set $V_n = U_1 \cup U_2 \cup \ldots \cup U_n$ and note that $V_n \cap Y = \tilde{U}_n$.

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots$$

is an ascending chain of open subsets of X. Since X is Noetherian there is an $n \in \mathbb{N}$ so that $V_n = V_{n+k}$ for all $k \in \mathbb{N}$. This implies that $\tilde{U}_n = V_n \cap Y = V_{n+k} \cap Y = \tilde{U}_{n+k}$ for all $k \in \mathbb{N}$ and Y is Noetherian.

(c) Since X is Noetherian, $X = X_1 \cup \ldots \cup X_n$ where X_i are the irreducible components of X. If X is a Noetherian Hausdorff space so is every X_i and we may assume that X is irreducible. We claim that $X = \{P\}$ is a one point space. Suppose that $P, Q \in X$ with $P \neq Q$. Then there are open subsets $U, V \subseteq X$ with $P \in U$, $Q \in V$ and $U \cap V = \emptyset$. But then $X = (X-U) \cup (X-V)$ with $X - U \neq X$ and $X - V \neq X$, contradicting that X is irreducible.

(3) [10pts] Let K be an infinite field, $f \in K[x_1, \ldots, x_n]$ a polynomial, and $\varphi_f : K^n \longrightarrow K$ the function defined by $\varphi_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$. Show that if φ_f is the zero function, then f is the zero polynomial.

Proof. The proof is by induction on the number of variables n. The case $n = 1$ is clear. For the induction step let $f \in K[x_1, \ldots, x_n]$ with $\varphi_f = 0$. Write

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{m} g_i(x_1, \ldots, x_{n-1}) x_n^i$$

where $g_i(x_1, \ldots, x_{n-1}) \in K[x_1, \ldots, x_{n-1}]$. For any $(a_1, \ldots, a_{n-1})$ consider the polynomial $f(a_1, \ldots, a_{n-1}, x_n) \in K[x_n]$. Obviously,

$$f(a_1, \ldots, a_{n-1}, x_n) = \sum_{i=0}^{m} g_i(a_1, \ldots, a_{n-1}) x_n^i$$

is a polynomial in one variable over K with $f(a_1, \ldots, a_{n-1}, b) = 0$ for all $b \in K$. Thus $g_i(a_1, \ldots, a_{n-1}) = 0$ for all $0 \leq i \leq m$ and all $(a_1, \ldots, a_{n-1}) \in K^{n-1}$. By induction hypothesis all $g_i(x_1, \ldots, x_{n-1})$ are the zero polynomials and hence $f = 0$.

(4) [12pts] Let K be a finite field. Show:

(a) For every $a = (a_1, \ldots, a_n) \in K^n$ there is a polynomial $f \in K[x_1, \ldots, x_n]$ with $f(a_1, \ldots, a_n) = 0$ and $f(b) \neq 0$ for all $b \in K^n - \{a\}$.

(b) For any function $\psi : K^n \longrightarrow K$ there is a polynomial $f \in K[x_1, \ldots . x_n]$ with $\psi = \varphi_f$.

(c) Any subset $V \subseteq K^n$ is the zero set of some polynomial $f \in K[x_1, \ldots, x_n]$.

Proof. (a) Let $K = \{c_1, \ldots, c_r\}$ and $a = (c_{i_1}, \ldots, c_{i_n})$. Define:

$$f(x_1, \ldots, x_n) = \frac{\prod_{j \neq i_1}(x_1 - c_j)}{\prod_{j \neq i_1}(c_{i_1} - c_j)} \frac{\prod_{j \neq i_2}(x_2 - c_j)}{\prod_{j \neq i_2}(c_{i_2} - c_j)} \cdots \frac{\prod_{j \neq i_n}(x_n - c_j)}{\prod_{j \neq i_n}(c_{i_n} - c_j)} - 1.$$

Then $f(a) = 0$ and $f(b) = 1$ for all $b \in K^n - \{a\}$. Note that the polynomial $g(x) = f(x) + 1$ has the property that $g(a) = 1$ and $g(b) = 0$ for all $b \in K^n - \{a\}$.
(b) Let $K^n = \{a_1, \ldots, a_s\}$ and for all $1 \leq i \leq s$ let $f_i \in K[x_1, \ldots, x_n]$ be a polynomial with $f_i(a_i) = 1$ and $f_i(b) = 0$ for all $b \in K^n - \{a_i\}$. If $\psi : K^n \longrightarrow K$ is a function set

$$f(x) = \sum_{i=1}^{s} \psi(a_i) f_i(x).$$

Then for all $a \in K^n$: $\psi(a) = f(a)$.
(c) If $V \subseteq K^n$ define a function $\psi : K^n \longrightarrow K$ by $\psi(a) = 0$ if $a \in V$ and $\psi(a) = 1$ if $a \notin V$. By (b) there is a polynomial $f \in K[x_1, \ldots, x_n]$ with $f(a) = \psi(a)$ for all $a \in K^n$. Thus $V = Z(f)$.

(5) [10pts] Let K be an algebraically closed field and $Y \subseteq \mathbb{A}_K^n$ an irreducible algebraic variety of dimension r. Let H be a hypersurface of $\mathbb{A}_K^n$ with $Y \not\subseteq H$. Show that every irreducible component of $Y \cap H$ has dimension $\leq r-1$.

Proof. We know that $Y = Z(P)$ where $P \subseteq K[x_1, \ldots, x_n]$ is a prime ideal. Since H is a hypersurface, $H = Z(f)$ for some $f \in K[x_1, \ldots, x_n]$ and $Y \not\subseteq H$ implies that $f \notin P$. Then

$$Y \cap H = Z(P) \cap Z(f) = Z(P + (f)).$$

If $P + (f) = k[x_1, \ldots, x_n]$, then $Y \cap H = \emptyset$ and $\dim(Y \cap H) \leq r - 1$. If $P + (f) \neq K[x_1, \ldots, x_n]$ then f is a nonzero nonunit in the domain $A(Y) = k[x_1, \ldots, x_n]/P$. Thus

$$\dim(K[x_1, \ldots, x_n]/(P + (f))) < \dim(k[x_1, \ldots, x_n]/P) = \dim(A(Y)).$$

In particular, $\dim(A(Y \cap H)) = \dim(K[x_1, \ldots, x_n]/(\text{rad}(P + (f))) \leq r - 1$.

(6) [10pts] Let R be a ring and $n \in \mathbb{N}$ an integer. Suppose that every ideal of R is generated by at most n elements. Show that $\dim(R) \leq 1$.

Proof. First note that R is a Noetherian ring. We need to show that for every prime ideal $P \subseteq R$, $\text{ht}P = \dim(R_P) \leq 1$. Since every ideal of R_P is extended from an ideal of R, we may assume that R is a local Noetherian ring with maximal ideal $\mathfrak{m}$ and that every ideal of R is generated by at most n elements. Let $P(t) \in \mathbb{Q}[t]$ be the Hilbert-Samuel polynomial of R with respect to the maximal ideal $\mathfrak{m}$, that is, for $s \in \mathbb{N}$ with $s \geq n_0$:

$$P(s) = \ell_R(R/\mathfrak{m}^{s+1}) = \sum_{i=0}^{s} \ell_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

Since $\ell_R(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ is the minimal number of generators of the ideal $\mathfrak{m}^i$, it follows that

$$P(s) \leq (s + 1)n$$

where n is a fixed integer. This implies that $\deg(P(t)) \leq 1$. Since $\deg(P(t)) = \dim(R)$ the assertion follows.

(7) [10pts] Let R be a ring so that for every maximal ideal $\mathfrak{m} \subseteq R$ the localization $R_\mathfrak{m}$ is Noetherian. Suppose that for every element $a \in R - (0)$ there are at most finitely many maximal ideals $\mathfrak{m} \subseteq R$ so that $a \in \mathfrak{m}$. Show that R is a Noetherian ring. Is the converse true?

Proof. Let $I \subseteq R$ be a nonzero ideal. Since every $a \in I - (0)$ is contained in at most finitely many maximal ideals, the ideal I is contained in at most finitely many maximal ideals. Suppose that $I \neq R$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be the maximal ideals containing I. Since $R_{\mathfrak{m}_i}$ is Noetherian for all $1 \leq i \leq s$ there are elements $a_1, \ldots, a_n \in I$ so that

$$IR_{\mathfrak{m}_i} = (a_1/1, \ldots, a_n/1)R_{\mathfrak{m}_i} \quad \text{for all} \quad 1 \leq i \leq s.$$

Let $J = (a_1, \ldots, a_n)$ be the ideal of R which is generated by the $a_i's$. Obviously, $J \subseteq I$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s, \mathfrak{m}_{s+1}, \ldots, \mathfrak{m}_t$ be the maximal ideals containing J. If $s = t$

then $I = J$ since $I_{\mathfrak{m}} = J_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subseteq R$. Suppose that $s < t$. Then for all $s+1 \le i \le t$ we have $I \not\subseteq \mathfrak{m}_i$. For all $s+1 \le i \le t$ take an element $b_i \in I - \mathfrak{m}_i$. We claim that

$$I = (a_1, \dots, a_n, b_{s+1}, \dots, b_t).$$

Let $K = (a_1, \dots, a_n, b_{s+1}, \dots, b_t)$ and let $\mathfrak{m} \subseteq R$ be a maximal ideal of R. If $\mathfrak{m} \neq \mathfrak{m}_i$ for all $1 \le i \le t$ then $I_{\mathfrak{m}} = K_{\mathfrak{m}} = R_{\mathfrak{m}}$, since $J \subseteq K$. If $\mathfrak{m} = \mathfrak{m}_i$ for some $s+1 \le i \le t$ then $I_{\mathfrak{m}} = R_{\mathfrak{m}} = K_{\mathfrak{m}}$ since $b_i \notin \mathfrak{m} = \mathfrak{m}_i$. If $\mathfrak{m} = \mathfrak{m}_i$ for some $1 \le i \le s$, then $I_{\mathfrak{m}} = J_{\mathfrak{m}} = K_{\mathfrak{m}}$ since $J \subseteq K \subseteq I$. Thus for all maximal ideals $\mathfrak{m} \subseteq R$ we have that $I_{\mathfrak{m}} = K_{\mathfrak{m}}$. By the local-global principle $I = K$.

(8) [14pts] Let K be a field and $T = K[\{x_i | i \in \mathbb{N}\}]$ the polynomial ring in infinitely many (countably) many variables over K. Let $\{n_i\}$ be a strictly increasing sequence of positive integers which satisfies the condition: $0 < n_i - n_{i-1} < n_{i+1} - n_i$ for all $i \in \mathbb{N}$. Consider the prime ideals $P_i = (x_j | n_i \le j < n_{i+1})$ in T and set $S = T - \cup_{i \in \mathbb{N}} P_i$ and $R = S^{-1}T$. Show

(a) The maximal ideals of R are exactly the ideals $S^{-1}P_i$ for all $i \in \mathbb{N}$.
(b) The ring $R_{S^{-1}P_i}$ is Noetherian of dimension $n_{i+1} - n_i$.
(c) R is a Noetherian ring of infinite dimension.

(This example is due to M. Nagata.)

Proof. (a) Let $i_{T,S} : T \longrightarrow R$ be the canonical map into the localization and $\mathfrak{m} \subseteq R$ be a maximal ideal of R. The preimage $i_{T,S}^{-1}(\mathfrak{m}) = P$ is a prime ideal of T with $P \cap S = \emptyset$. This implies that

$$P \subseteq \cup_{i \in \mathbb{N}} P_i.$$

Note that for every nonzero element $f \in T$ there is an integer $t \in \mathbb{N}$ so that $f \in K[x_1, \dots, x_t]$. If $f \in P$ is a nonzero element with $f \in K[x_1, \dots, x_t]$, then there is a maximal j so that $n_j \le t$. We claim that for this integer j:

$$(*) \quad P \subseteq \cup_{i=1}^{j} P_i.$$

Proof of (*). Suppose that there is an element $g \in P - \cup_{i=1}^{j} P_i$. Since $P \subseteq \cup_{i \in \mathbb{N}} P_i$, there is an $\ell > j$ so that $g \in P_\ell$. Then we can write

$$g = \sum a_\alpha m_\alpha$$

where $a_\alpha \in K - (0)$ and m_α monomials with the following property:

(i) For all α there is an i with $n_\ell \le i < n_{\ell+1}$ so that x_i divides m_α.
(ii) For all $1 \le k \le j$ there is an $m_{\alpha(k)}$ such that x_i does not divide $m_{\alpha(k)}$ for all $n_k \le i < n_{k+1}$.

If $f = \sum b_\beta n_\beta$ with $b_\beta \in K - (0)$ and n_β monomials, consider the sum:

$$f + g = \sum b_\beta n_\beta + \sum a_\alpha m_\alpha.$$

Since $f \in K[x_1, \dots, x_t]$ with $t < n_\ell$ the monomials n_β and m_α do not cancel each other. Furthermore $n_\beta \notin P_r$ for all $r > j$, $g \notin P_i$ for $i \le j$, and thus $f + g \notin P_i$ for all $i \in \mathbb{N}$, a contradiction. This proves the claim.

(*) implies that $P \subseteq P_i$ for some $i \in \mathbb{N}$ and hence $\mathfrak{m} = PR \subseteq P_iR$. Since $\mathfrak{m}$ is maximal $\mathfrak{m} = P_iR$ (and $P = P_i$).
(b)

$$R_{S^{-1}P_i} \cong T_{P_i} = K(x_j | j \in \mathbb{N} \text{ with } j < n_i \text{ or } j \geq n_{i+1})[x_{n_i}, \ldots, x_{n_{i+1}-1}]_{\tilde{P}_i}$$

where $\tilde{P}_i$ is the prime ideal generated by $x_{n_i}, \ldots, x_{n_{i+1}-1}$. This shows that $R_{S^{-1}P_i}$ is a Noetherian ring of dimension $n_{i+1} - n_i$.
(c) Let $d \in R - (0)$ be a nonunit of R. Then $d = f/g$ where $f, g \in T$ and $g \in S$. Since f is contained in only finitely many P_i the element d is contained in only finitely many maximal ideals of R. By Problem (7) the ring R is Noetherian. Since $\dim(R_{S^{-1}P_i}) = n_{i+1} - n_i$ and $n_{i+1} - n_i \to \infty$ if $i \to \infty$, the dimension of R is infinite.

(9) [16pts] Let K be a field, $R = K[x_1, \ldots, x_n]$ the polynomial ring over K, and $I \subseteq R$ an ideal. Show that;

$$\mathrm{ht}I + \dim(R/I) = \dim(R).$$

Proof. (a) We first show that we may assume that I is a prime ideal of R. Suppose that for every prime ideal $P \subseteq R$:

$$\mathrm{ht}P + \dim(R/P) = \dim(R) = n.$$

Let $I \subseteq R$ be an ideal and let $P \subseteq R$ be a prime ideal with $I \subseteq P$ and $\mathrm{ht}I = \mathrm{ht}P$. Assume that $\mathrm{ht}I + \dim(R/I) \neq n$. Since $\mathrm{ht}P + \dim(R/P) = n$, this implies that $\dim(R/I) > \dim(R/P)$. Let $Q \subseteq R$ be a prime ideal with $I \subseteq Q$ and $\dim(R/I) = \dim(R/Q)$. Since $\dim(R/Q) = n - \mathrm{ht}Q > \dim(R/P) = n - \mathrm{ht}P$ it follows that $\mathrm{ht}P > \mathrm{ht}Q$, a contradiction, since $\mathrm{ht}I = \inf\{\mathrm{ht}P \,|\, I \subseteq P \in \mathrm{Spec}(R)\}$.
(b) We claim that every maximal ideal of R has height n. The proof is by induction on n. The case $n = 1$ is trivial. Suppose that $n > 1$ and that $\mathfrak{m} \in R$ is a maximal ideal of R. Then $R/\mathfrak{m} = K[\alpha_1, \ldots, \alpha_n]$ is an algebraic field extension of K. By (3.1) $\mathfrak{m} = (f_1, \ldots, f_n)$ where $f_i \in K[x_1, \ldots, x_i]$ monic in x_i. Set $L = K[x_1]/(f_1)$, where f_1 is the minimal polynomial of α_1 over K. Then $\bar{\mathfrak{m}} = \mathfrak{m}/(f_1)$ is a maximal ideal of $L[x_2, \ldots, x_n]$ and by induction hypothesis $\mathrm{ht}\bar{\mathfrak{m}} = n-1$. Therefore $\mathrm{ht}\mathfrak{m} = n$.
(c) Let $P \subseteq R$ be a prime ideal. If P is maximal, then by (b):

$$\mathrm{ht}P + \dim(R/P) = n.$$

Suppose that $\dim(R/P) = r > 0$. The elements $x_1 + P, \ldots, x_n + P$ generate the quotient field $Q(R/P)$ over K. Moreover, $Q(R/P)$ has transcendence degree r over K and we may assume that $x_1 + P, \ldots, x_r + P$ is a transcendence basis of $Q(R/P)$ over K. This implies that $P \cap K[x_1, \ldots, x_r] = 0$. If Q is a prime ideal of R with $P \subseteq Q$ and $P \neq Q$, then

$$\dim(R/Q) < \dim(R/P).$$

Thus $Q(R/Q)$ has transcendence degree $< r$ over K. This implies that for all prime ideals Q with $P \subseteq Q$ and $P \neq Q$,

$$Q \cap K[x_1, \ldots, x_r] \neq 0$$

and with $S = K[x_1, \ldots, x_r] - (0)$ the ideal $PS^{-1}R$ is maximal in $S^{-1}R$. Note that

$$S^{-1}R = L[x_{r+1}, \ldots, x_n]$$

where $L = K(x_1, \ldots, x_r) = Q(K[x_1, \ldots, x_r])$. By (b),

$$\mathrm{ht}PS^{-1}R = \mathrm{ht}P = n - r.$$

This shows that $\mathrm{ht}P + \dim(R/P) = n$.