Solutions to Homework 2.

(1) [4pts] Let R be a ring and $Q \subseteq R$ an ideal with $\mathrm{rad}Q = P$ where $P \subseteq R$ is a prime ideal. Show that Q is P-primary if and only if for all $a, b \in R$ with $ab \in Q$ and $a \notin P$ we have that $b \in Q$.

Proof. Suppose that Q is P-primary. By (2.21) for all $a \in R$ the map $t_a : R/Q \longrightarrow R/Q$ is either injective or nilpotent. If $a \notin P = \mathrm{rad}Q$, the map t_a is not nilpotent, thus t_a is injective. Hence if $b \in R$ with $ab \in Q$, then $t_a(b) = 0$ and $b \in Q$. For the backward direction by (2.21) we have to show that for all $a \in R$, the map t_a is either injective or nilpotent. If t_a is not nilpotent then $a \notin P$ and for all $b \in R$ with $ab \in Q$ it follows that $b \in Q$. Hence t_a is injective.

(2) [10pts] Let R be a Noetherian ring, $P \subseteq R$ a prime ideal, and $i_{R,P} : R \longrightarrow R_P$ the canonical map into the localization. Define $P^{(n)} = i_{R,P}^{-1}(P^n R_P)$ and show:

(a) $P^{(n)}$ is a P-primary ideal.
(b) $P^{(n)}$ is the P-primary component of P^n.
(c) $P^{(n)} = P^n$ if and only if P^n is a primary ideal.

Proof.

Lemma. *Let $\varphi : R \longrightarrow T$ be a homomorphism of rings and let $Q' \subseteq T$ be a P'-primary ideal. Then $Q = \varphi^{-1}(Q')$ is a $P = \varphi^{-1}(P')$-primary ideal of R.*

Proof of Lemma. Let $a, b \in R$ with $ab \in Q$ and $a \notin Q$. Then $\varphi(a)\varphi(b) \in Q'$ and $\varphi(a) \notin Q'$. Since Q' is primary there is an $m \in \mathbb{N}$ so that $\varphi(b)^m \in Q'$. Thus $b^m \in Q$ and Q is a primary ideal. Since $\mathrm{rad}Q = P$ it follows that Q is P-primary.

(a) Since PR_P is the maximal ideal of R_P, the ideal $P^n R_P$ is PR_P-primary. The lemma shows that $P^{(n)} = i_{R,P}^{-1}(P^n R_P)$ is P-primary.
(b) Let $P^n = Q_1 \cap \ldots \cap Q_r$ be a shortest primary decomposition of P^n with Q_i a P_i-primary ideal. Since $\mathrm{rad}(P^n) = P = P_1 \cap \ldots \cap P_r$ the prime ideal $P = P_1$ is minimal in $\mathrm{Ass}(R/P^n)$. By Theorem (2.40) $Q_1 = i_{R,P}^{-1}(P^n R_P) = P^{(n)}$.
(c) follows from (a) and (b).

(3) [8pts] Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finite R-module, and $N = (0 :_M I) = \{m \in M \mid Im = 0\}$. Show that $\mathrm{Ass}_R(M/N) \subseteq \mathrm{Ass}_R(M)$.

Proof. Let $m \in M$ with $\mathrm{ann}(m + N) = P \in \mathrm{Ass}(M/N)$. We claim that $\mathrm{ann}(Im) = P$. Let $t \in R$ with $tIm = 0$, thus $tm \in N$ and $t \in \mathrm{ann}(m + N) = P$. Conversely, if $p \in P$, then $pm \in N$ and $I(pm) = 0$ by definition of N. Hence $p \in \mathrm{ann}(Im)$.

This shows that $\mathrm{Supp}_R(Im) = V(\mathrm{ann}(Im)) = V(P)$. Therefore P is a minimal prime ideal of $\mathrm{Supp}_R(Im)$ and hence $P \in \mathrm{Ass}(Im)$. Since Im is an R-submodule of M, the assertion follows.

(4) [10pts] Let R be a ring and M an R-module.

(a) For every prime ideal $P \subseteq R$ show that $P \in \mathrm{Supp}_R(M)$ if and only if there is a submodule $N \subseteq M$ with $P \in \mathrm{Ass}_R(M/N)$.
(b) Suppose that R is Noetherian and that M is finite. Show that for every prime ideal $P \in \mathrm{Supp}_R(M)$ there is a normal series $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that for all $0 \leq i \leq n-1$, $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \mathrm{Spec}(R)$ and with $P_j = P$ for some $0 \leq j \leq n-1$.

Proof. (a) For the backward direction let $P \in \mathrm{Ass}_R(M/N)$ where $N \subseteq M$ is a submodule. Then $P \in \mathrm{Supp}_R(M/N)$ and $(M/N)_P \neq 0$, hence $M_P \neq 0$.

For the forward direction let $P \in \mathrm{Supp}_R(M)$. Then $M_P \neq 0$ and there is an $m \in M$ with $m/1 \neq 0$ in M_P. Set $N = Pm$. We claim that $P = \mathrm{ann}(m + N)$ in M/N. Obviously, $P \subseteq \mathrm{ann}(m + N)$. If $b \in \mathrm{ann}(m + N)$, then there is a $p \in P$ with $bm = pm$ and therefore $b - p \in \mathrm{ann}(m) \subseteq P$. This shows that $P = \mathrm{ann}(m + N) \in \mathrm{Ass}_R(M/N)$.
(b) Piece together a normal series of N and a normal series of M/N where $P \in \mathrm{Supp}(M/N)$.

(5) [10pts] Let R be a Noetherian ring, M a finite R-module and $I = \mathrm{ann}(M)$.

(a) If I is a prime ideal of R, then $I \in \mathrm{Ass}(M)$.
(b) $\mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(M)$.
(c) If $P, Q \subseteq R$ are distinct prime ideals with $P \subset Q$ and $M = R/P \oplus R/Q$, then $\mathrm{Ass}_R(R/I) \neq \mathrm{Ass}_R(M)$.

Proof. (a) If $I = \mathrm{ann}(M)$ is prime then I is the minimal prime ideal in $\mathrm{Supp}_R(M) = V(\mathrm{ann}(M))$. Thus by (2.15) $I \in \mathrm{Ass}_R(M)$.
(b) Let $P \in \mathrm{Ass}_R(R/I)$ and $a \in R$ with $\mathrm{ann}(a + I) = P$. We claim that $P = \mathrm{ann}(aM)$. Obviously, $P \subseteq \mathrm{ann}(aM)$. Let $t \in R$ with $t(aM) = 0$. Then $ta \in I$ and $t \in \mathrm{ann}(a + I) = P$. Hence $\mathrm{ann}(aM) = P$. By (a) $P \in \mathrm{Ass}_R(aM) \subseteq \mathrm{Ass}_R(M)$.
(c) If $M = R/P \oplus R/Q$, then $\mathrm{ann}(M) = P \cap Q = P$, since $P \subset Q$. Thus $P \in \mathrm{Ass}_R(M)$ by (a). Since $\mathrm{ann}((0, 1)) = Q$, $Q \in \mathrm{Ass}_R(M) \neq \mathrm{Ass}_R(R/P) = \{P\}$.

(6) [8pts] Let R be a Noetherian ring, $P \subseteq R$ a prime ideal, and M a finite R-module.

(a) If $N \subseteq M$ is a P-primary submodule of M, then there is an $n \in \mathbb{N}$ with $P^n M \subseteq N$. The smallest $n \in \mathbb{N}$ with $P^n M \subseteq N$ is called the *exponent* of N in M and is denoted by $e(M/N)$.
(b) Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a set of P-primary submodules of M. Then $N = \cap_{\lambda \in \Lambda} N_\lambda$ is P-primary if and only if the set $\{e(M/N_\lambda)\}_{\lambda \in \Lambda}$ is bounded above. In this case $e(M/N) \geq e(M/N_\lambda)$ for all $\lambda \in \Lambda$.

Proof. (a) If $N \subseteq M$ is P-primary then $\mathrm{Ass}(M/N) = \{P\}$. Moreover, $\mathrm{rad}(\mathrm{ann}(M/N)) = P$ and there is an $n \in \mathbb{N}$ with $P^n \subseteq \mathrm{ann}(M/N)$, since R is a Noetherian ring.
(b) If N is P-primary then by (a) there is an $n \in \mathbb{N}$ with $P^n M \subseteq N \subseteq N_\lambda$. Thus $e(M/N_\lambda) \leq n$ for all λ.

For the backward direction suppose that $e(M/N_\lambda) \leq t$ for some $t \in \mathbb{N}$ and all λ. Then $P^t M \subseteq N$ and $\mathrm{Supp}(M/N) \subseteq V(P)$. In particular, for all $Q \in \mathrm{Ass}_R(M/N)$: $P \subseteq Q$. Let $Q \in \mathrm{Ass}_R(M/N)$ and $m \in M$ with $\mathrm{ann}(m+N) = Q$. Since $m+N \neq 0$ in M/N there is a λ with $m \notin N_\lambda$. Since $Q \subseteq \mathrm{ann}(m + N_\lambda)$ and N_λ P-primary it follows that $\mathrm{ann}(m + N_\lambda) = Q = P$. The last statement is trivial.

(7) [12pts] Let K be a field and $R = K[x, y, z]/(z^2 - xy)$ where x, y, z are variables over K. Show that $P = (x, z)R$ is a prime ideal of R, P^2 is not primary, and $P^2 = Q_1 \cap Q_2$ is a primary decomposition of P^2 where $Q_1 = Rx$ and $Q_2 = m^2$ with $m = (x, y, z)R$.

Proof. Obviously,

$$R/P = R/(x, z) \cong K[x, y, z]/(z^2 - xy, x, z) = K[x, y, z]/(x, z) \cong K[y]$$

and P is a prime ideal of R.

In order to show that $P^2 = Q_1 \cap Q_2$ in R with $Q_1 = Rx$ and $Q_2 = (x,y,z)^2R$, set $A = K[x,y,z]$ and consider preimages in A:

$$\begin{aligned}\tilde{P^2} &= (x^2, xz, z^2, z^2 - xy) = (x^2, xz, z^2, xy)\\ \tilde{Q_1} &= (x, z^2 - xy) = (x, z^2)\\ \tilde{Q_2} &= (x^2, xy, xz, y^2, yz, z^2).\end{aligned}$$

Then

$$(x, z^2) \cap (x^2, xy, xz, y^2, yz, z^2) = (x^2, xy, xz, z^2) = \tilde{P^2}.$$

Thus $P^2 = Q_1 \cap Q_2$.

$Q_1 = Rx$ is P-primary in R if and only if $\tilde{Q}_1 = (x, z^2)$ is $\tilde{P} = (x,z)$-primary in A. Obviously, $\mathrm{rad}(x, z^2) = (x,z)$. Note that $(x,z^2)A_{(x,z)}$ is primary with respect to the maximal ideal and that the ideal $i^{-1}_{A,(x,z)}(x, z^2)$ is (x,z)-primary by the Lemma of problem 2 (where $i_{A,(x,z)} : A \longrightarrow A_{(x,z)}$ is the natural map). Thus it suffices to show that $i^{-1}_{A,(x,z)}(x,z^2) = (x,z^2)$. Let $f \in i^{-1}_{A,(x,z)}(x,z^2)$. Since $x, z^2 \in (x, z^2)$ we may assume that

$$f = \sum a_i y^i + \sum b_j y^j z$$

with $a_i, b_j \in K$. Then there is a $g \in A - (x,z)$ with $fg \in (x,z^2)A$. Again we may assume that g is of the form:

$$g = \sum c_i y^i + \sum d_j y^j z$$

where $c_i, d_j \in K$. Thus

$$\begin{aligned}fg &= (\sum a_i y^i + \sum b_j y^j z)(\sum c_i y^i + \sum d_j y^j z)\\ &\equiv (\sum a_i y^i)(\sum c_i y^i) + (\sum b_j y^j z)(\sum c_i y^i) + (\sum a_i y^i)(\sum d_j y^j z) \bmod (x, z^2).\end{aligned}$$

First note that $\sum c_i y^i \neq 0$ since $g \notin (x,z)$. Since no term of $(\sum a_i y^i)(\sum c_i y^i)$ cancels with any other term of the sum, it follows that $\sum a_i y^i = 0$ implying that then also $\sum b_j y^j z = 0$. Hence $f \in (x,z^2)$ and (x,z^2) is P-primary (in A). It is easy to see that $P^2 = Q_1 \cap Q_2$ is a shortest primary decomposition. By (2.27) $\mathrm{Ass}_R(R/P^2) = \mathrm{Ass}_R(R/Q_1) \cup \mathrm{Ass}_R(R/Q_2) = \{P, m\}$ and P^2 is not primary.

(8) [10pts] Let R be a Noetherian ring and $a \in R$ a NZD of R. Show that $\mathrm{Ass}_R(R/(a)) = \mathrm{Ass}_R(R/(a^n))$ for all $n \in \mathbb{N}$.

Proof. Since $a \in R$ is a NZD, the R-linear map:

$$\varphi : R/(a^{n-1}) \longrightarrow aR/(a^n)$$

defined by $\varphi(x + (a^{n-1})) = ax + (a^n)$ is an isomorphism of R-modules. From the short exact sequence;

$$0 \longrightarrow R/(a^{n-1}) \xrightarrow{\varphi} R/(a^n) \to R/(a) \to 0$$

we obtain that

$$\mathrm{Ass}_R(R/(a^n)) \subseteq \mathrm{Ass}_R(R/(a^{n-1})) \cup \mathrm{Ass}_R(R/(a)).$$

The proof is by induction on n. Since

$$\mathrm{Ass}_R(R/(a^n)) \subseteq \mathrm{Ass}_R(R/(a^{n-1})) \cup \mathrm{Ass}_R(R/(a))$$

and

$$\mathrm{Ass}_R(R/(a^{n-1})) = \mathrm{Ass}_R(aR/(a^n)) \subseteq \mathrm{Ass}_R(R/(a^n))$$

the induction hypothesis

$$\mathrm{Ass}_R(R/(a^{n-1})) = \mathrm{Ass}_R(R/(a))$$

yields that

$$\mathrm{Ass}_R(R/(a^n)) = \mathrm{Ass}_R(R/(a)).$$

(9) [12pts] Let R be a ring and x a variable over R. Show:

(a) $\mathrm{Ass}_{R[x]}(R[x]) = \{PR[x] \mid P \in \mathrm{Ass}_R(R)\}$.

(b) If $Q \subseteq R$ is P-primary, then $QR[x]$ is $PR[x]$-primary.

Proof. (a) Let $P \in \mathrm{Ass}_R(R)$ and $a \in R$ with $\mathrm{ann}_R(a) = P$. Then $\mathrm{ann}_{R[x]}(a) = PR[x]$ and $PR[x] \in \mathrm{Ass}_{R[x]}(R[x])$. Conversely, suppose that $Q \in \mathrm{Ass}_{R[x]}(R[x])$ with $P = R \cap Q$. If $Q \neq PR[x]$ let $f(x) \in R[x]$ be a polynomial of minimal degree with $\mathrm{ann}_{R[x]}(f(x)) = Q$. Write $f(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n \neq 0$ and $\deg(f(x)) = n$. Then for all $0 \leq i \leq n$, $P \subseteq \mathrm{ann}_R(a_i)$ and for some $1 \leq i \leq n$ $P = \mathrm{ann}_R(a_i)$. Let $g(x) \in Q - PR[x]$. We may assume that $g(x) = \sum_{i=0}^{m} b_i x^i$ where $b_i \in R - P$ or $b_i = 0$ and $b_m \neq 0$. Then $b_m a_n = 0$ and $b_m f(x) = 0$ or $\deg(b_m f(x)) < n$. If $b_m f(x) \neq 0$ we claim that $\mathrm{ann}_{R[x]}(b_m f(x)) = Q$. One inclusion is trivial. For the other suppose that $h(x) \in \mathrm{ann}_{R[x]}(b_m f(x))$. Then $h(x) b_m f(x) = 0$ and $h(x) b_m \in \mathrm{ann}_{R[x]}(f(x)) = Q$. Since $b_m \notin Q$, $h(x) \in Q$ and therefore $\mathrm{ann}_{R[x]}(b_m f(x) = Q$, contradicting the minimality of the degree of $f(x)$. Thus $b_m f(x) = 0$ and $b_m \in Q \cap R = P$, another contradiction.

(b) Apply (a) to the ring R/Q.

(10) [16pts] Let R be a ring and M a finite R-module. Consider the set of submodules of M defined by $\Lambda = \{IM \mid I \subseteq R \quad \text{an ideal}\}$ and suppose that Λ satisfies the a.c.c. and the d.c.c. Show that M is an Artinian R-module.

Proof. We consider M as an $(R/\mathrm{ann}(M))$-module and assume that $\mathrm{ann}(M) = (0)$.

Claim (a) : R is a semilocal ring.

Proof (a). Consider the following subset of Λ:

$$\Gamma_1 = \{JM \mid J \text{ is a finite product of maximal ideals of } R\}.$$

By assumption Γ_1 has a minimal element JM where $J = \mathfrak{m}_1 \ldots \mathfrak{m}_n$. If $\mathfrak{w} \in$ m-Spec(R) is a maximal ideal different from the $\mathfrak{m}_i$ then by the minimality of JM: $JM = \mathfrak{w}JM$. Since $JR_{\mathfrak{w}} = R_{\mathfrak{w}}$, we have that $M_{\mathfrak{w}} = \mathfrak{w}M_{\mathfrak{w}}$. By Nakayama's Lemma $M_{\mathfrak{w}} = 0$. Using again that M is finitely generated we see that there is an element $t \in R - \mathfrak{w}$ with $tM = 0$, a contradiction to $\mathrm{ann}(M) = 0$.

Claim (b) : For every ideal $I \subseteq R$ the R-module IM is finite.

Proof (b). Let $I \subseteq R$ be an ideal. Consider the set

$$\Gamma_2 = \{JM \mid J \subseteq I \text{ and } JM \text{ is a finite } R\text{-module}\}.$$

By assumption Γ_2 has a maximal element J_0M. If $x \in IM$ then there are finite many elements $a_1, \ldots, a_n \in I$ and $m_1, \ldots, m_n \in M$ with $x = \sum_{i=1}^n a_i m_i \in (J_0 + (a_1, \ldots, a_n))M$ and by the maximality of J_0M we have that $x \in J_0M$. Thus $J_0M = IM$ and IM is a finite R-module.

(c) Suppose that m-Spec$(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. By the a.c.c. there is an integer $k \in \mathbb{N}$ so that $(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)^k M = (\mathfrak{m}_1, \ldots, \mathfrak{m}_n)^{k+1} M$. By (b) the R-module $(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)^k M$ is finitely generated and thus by Nakayama $(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)^k M = 0$.

(d) Set $I_{(k_1,\ldots,k_n)} = \mathfrak{m}_1^{k_1} \mathfrak{m}_2^{k_2} \ldots \mathfrak{m}_n^{k_n}$ and consider the chain of submodules:

$$M \supseteq I_{(1,0,\ldots,0)}M \supseteq \ldots \supseteq I_{(k,0,\ldots,0)}M \supseteq I_{(k,1,0,\ldots,0)}M \supseteq \ldots \supseteq I_{(k,k,\ldots,k)}M.$$

Each factor module

$$N_{(k,\ldots,k,t,0,\ldots,0)} = \mathfrak{m}_1^k \ldots \mathfrak{m}_i^k \mathfrak{m}_{i+1}^t M / \mathfrak{m}_1^k \ldots \mathfrak{m}_i^k \mathfrak{m}_{i+1}^{t+1} M$$

is a finite vector space over $K_{i+1} = R/\mathfrak{m}_{i+1}$. Thus every $N_{(k,\ldots,k,t,0,\ldots,0)}$ is an R-module of finite length and M is an R-module of finite length.