COMMUTATIVE ALGEBRA
HOMEWORK 7, DUE 4-4-12

(1) Let k be a field, $R = k[x, y, z]$ the polynomial ring in 3 variables over k. Show:

(a) [8pts] $x(x-1), xy-1, xz$ is a regular sequence in R.
(b) [4pts] $x(x-1), xz, xy-1$ is *not* a regular sequence in R.

(2) [8pts] Let k be a field, $R = k[x, y]_{(x,y)}$ the localized polynomial ring, and $S = R[z]/(xz, yz, z^2)$, where z is a variable over R. Determine $\dim(S)$ and $\operatorname{depth}(S)$.

(3) [16pts] Let k be a field and x_1, x_2, x_3, x_4 variables over k. Show that the ring $R = (k[x_1, x_2, x_3, x_4]/(x_1x_3, x_1x_4, x_2x_3, x_2x_4))_{(x_1,x_2,x_3,x_4)}$ is not a CM-ring.

Definition. *Let R be a ring and M an R-module. A sequence $a_\bullet = a_1, \ldots, a_n \in R$ is called weakly M-regular or a weak M-sequence, if for all $1 \leq i \leq n$ the element a_i is a nonzero divisor in $M/(a_1, \ldots, a_{i-1})M$.*

(4) [12pts] Let R be a ring, M an R-module, and $a_\bullet$ a weakly M-regular sequence. Let

$$N_2 \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence of R-modules. Show that the induced sequence

$$N_2/(a_\bullet)N_2 \longrightarrow N_1/(a_\bullet)N_1 \longrightarrow N_0/(a_\bullet)N_0 \longrightarrow M/(a_\bullet)M \longrightarrow 0$$

is exact.

(5) [12pts] Let R be a ring and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

an exact sequence of R-modules. Suppose that the sequence $a_\bullet$ is a weakly regular for M' and M''. Show that $a_\bullet$ is weakly M-regular.

(6) [10pts] Extend the result of problem (4) to show: Let R be a ring, and $a_\bullet = a_1, \ldots, a_n \in R$ a sequence of elements in R. If the complex

$$N_\bullet : \ldots \to N_m \xrightarrow{\varphi_m} N_{m-1} \to \ldots \to N_0 \to N_{-1} \to 0$$

is exact and $a_\bullet$ is weakly N_i-regular for all i, then the complex $N_\bullet \otimes_R R/(a_\bullet)$ is exact.

(7) Let R be a ring, M an R-module.

(a) [8pts] Prove that if $a_\bullet$ is weakly M-regular then $\mathrm{Tor}_1^R(M, R/(a_\bullet)) = 0$.

(b) [2pts] If, in addition, $a_\bullet$ is a weak R-sequence, prove that $\mathrm{Tor}_i^R(M, R/(a_\bullet)) = 0$ for all $i \geq 1$.

(8) [20pts] Let R be a ring, M an R-module, $a_1, a_2, \ldots, a_n \in R$, and set $I = (a_1, \ldots, a_n) \subseteq R$. Prove that the sequence $a_1, \ldots, a_n$ is M-quasi regular if and only if $a_1 + I^2, \ldots, a_n + I^2 \in I/I^2$ is $\mathrm{gr}_I(M)$-regular.