Homework 6 (due: 3-14-12).

(1) Let K be a field, x a variable over K, and $R = K[x]_{(x)}$. Find minimal projective resolutions of

(a) [3pts] the R-module $R/(x)$

(b) [5pts] the $R/(x^n)$-module $R/(x)$ where $n \geq 3$.

(2) [6pts] Let M a $\mathbb{Z}$-module. Determine projdim(M).

(3) Let G be an abelian group and $n \in \mathbb{Z}$. Show:

(a) [5pts]

$$\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, G) = G[n] = \{x \in G \mid nx = 0\}$$

(b) [3pts] $\mathrm{Tor}_1(\_, G)$ is left exact.

(4) [8pts] Let R be a ring and $I, J \subseteq R$ ideals. Show:

$$\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/IJ.$$

(5) Let R be a ring, $x \in R$ a NZD, non-unit, and M an R-module.

(a) [10pts] Determine $\mathrm{Ext}^i_R(R/(x), M)$ for all $i \geq 0$.

(b) [2pts] Determine $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all $n, m \in \mathbb{Z}$ and all $i \geq 0$.

(6) [10pts]Let R be a ring and $I \subseteq R$ an ideal. Show:

$$\mathrm{Ext}^1_R(R/I, R/I) = \mathrm{Hom}_R(I/I^2, R/I) = \mathrm{Hom}_R(\mathrm{Tor}^R_1(R/I, R/I), R/I).$$

(7) Let R be a Noetherian ring and $P \subseteq R$ a prime ideal.

(a) [8pts] If E is an injective R-module show that E_P is both R_P-injective and R-injective.

(b) [6pts] Let M be an R-module and $E(M)$ the injective hull of M. Then $E(M)_P$ is the injective hull of the R_P-module M_P.

(c) [4pts] Let $E^\bullet$ be a minimal injective resolution of an R-module M. Show that $E^\bullet_P$ is a minimal injective resolution of the R_P-module M_P.

(8) [16pts] Let M be an R-module. Show that the following are equivalent:

(a) M is a flat R-module.

(b) For every ideal $I \subseteq R$ the canonical morphism:

$$I \otimes_R M \longrightarrow IM$$

is injective.

(c) For every finitely generated ideal $I \subseteq R$ the canonical morphism:

$$I \otimes_R M \longrightarrow IM$$

is injective.

(9) Let R be a ring, $S = R[x_1, \ldots, x_n]$ the polynomial ring over R in n variables, and let

$$f = \sum_{(i) \in \mathbf{N}^n} a_{(i)} x_1^{i_1} \ldots x_n^{i_n}$$

be a polynomial in S. Put $T = S/(f)$. Show:

(a) [10pts] If $(a_{(i)})_{(i) \in \mathbf{N}^n} = R$ then T is flat over R.

(b) [4pts] If $(a_{(i)})_{(i) \in \mathbf{N}^n - (0)} = R$ then T is faithfully flat over R.