Homework 5 (due: 2-15-12).

(1) Prove the *Five Lemma:*
Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

and prove:

(a) [4pts] If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
(b) [4pts] If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.
(c) [2pts] If t_1, t_2, t_4 and t_5 are isomorphisms, then t_3 is an isomorphism.

(2)[10pts] Let R be a commutative ring with $1 \neq 0$ and let P and Q be projective R-modules. Show that $Q \otimes_R P$ is a projective R-module.

(3)[10pts] Prove that every projective R-module has a free complement, i.e., there is a free R-module F so that $P \oplus F$ is a free R-module.

(4)[12pts] Let R be a Noetherian ring, M be a finite R-module, and $\{N_i\}_{i \in I}$ a set of R-modules. Show:

$$\mathrm{Hom}_R(M, \oplus_{i \in I} N_i) \cong \oplus_{i \in I} \mathrm{Hom}_R(M.N_i).$$

(5)[10pts] Let R be a commutative ring and M an R-module. Suppose that

$$0 \to K_1 \to P_1 \to M \to 0 \quad \text{and} \quad 0 \to K_2 \to P_2 \to M \to 0$$

are exact sequences with projective modules P_1 and P_2. Show that $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

(6) Let R be a commutative domain, K its field of quotients. Prove:

(a) [10pts] A torsion-free R-module M is injective if and only if M is divisible.
(b) [3pts] K is the injective hull of A.

(7)[10pts] Let R be a Noetherian ring. Show that a direct sum of injective R-modules is an injective R-module.

(8)[10pts] Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local homomorphism of local Noetherian rings, and let N be an R-flat S-module such that $N/\mathfrak{m}N$ has finite length (as an S-module). Show that for every finite length R-module M:

$$l_S(M \otimes_R N) = l_R(M) l_S(N/\mathfrak{m}N).$$

(9)[15pts] Let A be a commutative local ring,

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow 0$$

an exact sequence of finitely generated free A-modules. Prove that

$$\sum_{i=0}^{n} (-1)^i \mathrm{rk}(F_i) = 0$$

where for a finitely generated free A-module F $\mathrm{rk}(F)$ denotes the number of elements in a basis of F.