Homework 4 (due: 12-9-11).

(1) [10pts] Let R be a semilocal Noetherian ring and $I \subseteq R$ an ideal of R. Show that the following conditions are equivalent:

(a) I is an ideal of definition of R.
(b) $I \subseteq \mathrm{Jrad}(R)$ and R/I is an Artinian ring.
(c) $I \subseteq \mathrm{Jrad}(R)$ and R/I has finite length.
(d) $\mathrm{Supp}_R(R/I) = \mathrm{mSpec}(R)$.

(2) [12pts] Let R be a discrete valuation domain. Show that the polynomial ring $R[x]$ has maximal ideals of height one and of height two.

(3) [8pts] Let $R \subseteq S$ be an extension of rings such that the set $S - R$ is closed under multiplication. Show that R is integrally closed in S.

(4) [10pts] Let R be a normal domain, $K = Q(R)$ its field of quotients, and $f(x) \in R[x]$ a monic polynomial. Show that $f(x)$ is irreducible in $K[x]$ if and only if $f(x)$ is irreducible in $R[x]$.

(5) Let $R \subseteq S$ be an extension of rings with S integral over R. Show:

(a) [4pts] If $a \in R$ is a unit in S then a is a unit in R.
(b) [8pts] The Jacobson radical of R is the contraction of the Jacobson radical of S.

(6) [10pts] Let $R \subseteq S$ be an extension of rings with S finitely generated and integral over R. Show that for every prime ideal $P \subseteq R$ there are only finitely many prime ideals $Q \subseteq S$ which lie over P.

(7) [20pts] Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial and let $Y = Z(f)$ be the algebraic variety defined by f. Y is called *non-singular* or *smooth* at a point $P \in Y$ if not all of the partial derivatives $\partial f/\partial x_i$ vanish at P. Let $A(Y)$ be the coordinate ring of Y and let $\mathfrak{m}_P \subseteq A(Y)$ be the maximal ideal of $A(Y)$ corresponding to P (that is, if $P = (a_1, \ldots, a_n)$, then $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)/(f))$. Show that Y is smooth at P if and only if the ring $A(Y)_{\mathfrak{m}_P}$ is regular.

(8) [18pts] Let $K \subseteq L$ be an extension of fields, $Q \subseteq L[x_1, \ldots, x_n]$ a prime ideal in the polynomial ring in n variables over L, and $P = Q \cap K[x_1, \ldots, x_n]$ its contraction to the polynomial ring over K. Show that $\mathrm{ht}Q \geq \mathrm{ht}P$ and that equality holds if L is algebraic over K. Use this to show that if two polynomials $f, g \in K[x_1, \ldots, x_n]$ have no common divisor in $K[x_1, \ldots, x_n]$ then f and g have no common divisor in $L[x_1, \ldots, x_n]$.