Homework 3 due: 11-18-11.

(1) [6pts] For a polynomial $P(t) \in \mathbb{Q}$ show that the following conditions are equivalent:

(a) $P(n) \in \mathbb{Z}$ for all integers $n \in \mathbb{Z}$.
(b) $P(n) \in \mathbb{Z}$ for all but finitely many integers $n \in \mathbb{Z}$.
(c) $P(t) = \sum_{i=0}^{n} a_i \binom{t}{i}$ with $a_i \in \mathbb{Z}$ and $n \in \mathbb{N}$ suitable.

(2) [12pts] Show:

(a) A Noetherian topological space is quasi-compact, that is, every open cover has a finite subcover.
(b) Any subset of a Noetherian topological space is Noetherian.
(c) A Hausdorff Noetherian space is a finite set with the discrete topology.

(3) [10pts] Let K be an infinite field, $f \in K[x_1, \ldots, x_n]$ a polynomial, and $\varphi_f : K^n \longrightarrow K$ the function defined by $\varphi_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$. Show that if φ_f is the zero function, then f is the zero polynomial.

(4) [12pts] Let K be a finite field. Show:

(a) For every $a = (a_1, \ldots, a_n) \in K^n$ there is a polynomial $f \in K[x_1, \ldots, x_n]$ with $f(a_1, \ldots, a_n) = 0$ and $f(b) \neq 0$ for all $b \in K^n - \{a\}$.
(b) For any function $\psi : K^n \longrightarrow K$ there is a polynomial $f \in K[x_1, \ldots .x_n]$ with $\psi = \varphi_f$.
(c) Any subset $V \subseteq K^n$ is the zero set of some polynomial $f \in K[x_1, \ldots, x_n]$.

(5) [10pts] Let K be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an irreducible algebraic variety of dimension r. Let H be a hypersurface of $\mathbb{A}^n_K$ with $Y \nsubseteq H$. Show that every irreducible component of $Y \cap H$ has dimension $\leq r - 1$.

(6) [10pts] Let R be a ring and $n \in \mathbb{N}$ an integer. Suppose that every ideal of R is generated by at most n elements. Show that $\dim(R) \leq 1$.

(7) [10pts] Let R be a ring so that for every maximal ideal $\mathfrak{m} \subseteq R$ the localization $R_\mathfrak{m}$ is Noetherian. Suppose that for every element $a \in R - (0)$ there are at most finitely many maximal ideals $\mathfrak{m} \subseteq R$ so that $a \in \mathfrak{m}$. Show that R is a Noetherian ring. Is the converse true?

(8) [14pts] Let K be a field and $T = K[\{x_i | i \in \mathbb{N}\}]$ the polynomial ring in infinitely many (countably) many variables over K. Let $\{n_i\}$ be a strictly increasing sequence of positive integers which satisfies the condition: $0 < n_i - n_{i-1} < n_{i+1} - n_i$ for all $i \in \mathbb{N}$. Consider the prime ideals $P_i = (x_j | n_i \leq j < n_{i+1})$ in T and set $S = T - \cup_{i \in \mathbb{N}} P_i$ and $R = S^{-1}T$. Show

(a) The maximal ideals of R are exactly the ideals $S^{-1}P_i$ for all $i \in \mathbb{N}$.
(b) The ring $R_{S^{-1}P_i}$ is Noetherian of dimension $n_{i+1} - n_i$.
(c) R is a Noetherian ring of infinite dimension.

(This example is due to M. Nagata.)

(9) [16pts] Let K be a field, $R = K[x_1, \ldots, x_n]$ the polynomial ring over K, and $I \subseteq R$ an ideal. Show that;

$$\mathrm{ht} I + \dim(R/I) = \dim(R).$$