Homework 2 (due: 10-28-11).

(1) [4pts] Let R be a ring and $Q \subseteq R$ an ideal with rad$Q = P$ where $P \subseteq R$ is a prime ideal. Show that Q is P-primary if and only if for all $a, b \in R$ with $ab \in Q$ and $a \notin P$ we have that $b \in Q$.

(2) [10pts] Let R be a Noetherian ring, $P \subseteq R$ a prime ideal, and $i_{R,P} : R \longrightarrow R_P$ the canonical map into the localization. Define $P^{(n)} = i_{R,P}^{-1}(P^n R_P)$ and show:

(a) $P^{(n)}$ is a P-primary ideal.
(b) $P^{(n)}$ is the P-primary component of P^n.
(c) $P^{(n)} = P^n$ if and only if P^n is a primary ideal.

(3) [8pts] Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finite R-module, and $N = (0 :_M I) = \{m \in M \mid Im = 0\}$. Show that $\mathrm{Ass}_R(M/N) \subseteq \mathrm{Ass}_R(M)$.

(4) [10pts] Let R be a ring and M an R-module.

(a) For every prime ideal $P \subseteq R$ show that $P \in \mathrm{Supp}_R(M)$ if and only if there is a submodule $N \subseteq M$ with $P \in \mathrm{Ass}_R(M/N)$.
(b) Suppose that R is Noetherian and that M is finite. Show that for every prime ideal $P \in \mathrm{Supp}_R(M)$ there is a normal series $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that for all $0 \leq i \leq n-1$, $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \mathrm{Spec}(R)$ and with $P_j = P$ for some $0 \leq j \leq n-1$.

(5) [10pts] Let R be a Noetherian ring, M a finite R-module and $I = \mathrm{ann}(M)$.

(a) If I is a prime ideal of R, then $I \in \mathrm{Ass}(M)$.
(b) $\mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(M)$.
(c) If $P, Q \subseteq R$ are distinct prime ideals with $P \subset Q$ and $M = R/P \oplus R/Q$, then $\mathrm{Ass}_R(R/I) \neq \mathrm{Ass}_R(M)$.

(6) [8pts] Let R be a Noetherian ring, $P \subseteq R$ a prime ideal, and M a finite R-module.

(a) If $N \subseteq M$ is a P-primary submodule of M, then there is an $n \in \mathbb{N}$ with $P^n M \subseteq N$. The smallest $n \in \mathbb{N}$ with $P^n M \subseteq N$ is called the *exponent* of N in M and is denoted by $e(M/N)$.
(b) Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a set of P-primary submodules of M. Then $N = \cap_{\lambda \in \Lambda} N_\lambda$ is P-primary if and only if the set $\{e(M/N_\lambda)\}_{\lambda \in \Lambda}$ is bounded above. In this case $e(M/N) \geq e(M/N_\lambda)$ for all $\lambda \in \Lambda$.

(7) [12pts] Let K be a field and $R = K[x, y, z]/(z^2 - xy)$ where x, y, z are variables over K. Show that $P = (x, z)R$ is a prime ideal of R, P^2 is not primary, and $P^2 = Q_1 \cap Q_2$ is a primary decomposition of P^2 where $Q_1 = Rx$ and $Q_2 = m^2$ with $m = (x, y, z)R$.

(8) [10pts] Let R be a Noetherian ring and $a \in R$ a NZD of R. Show that $\mathrm{Ass}_R(R/(a)) = \mathrm{Ass}_R(R/(a^n))$ for all $n \in \mathbb{N}$.

(9) [12pts] Let R be a ring and x a variable over R. Show:

(a) $\mathrm{Ass}_{R[x]}(R[x]) = \{PR[x] \mid P \in \mathrm{Ass}_R(R)\}$.
(b) If $Q \subseteq R$ is P-primary, then $QR[x]$ is $PR[x]$-primary.

(10) [16pts] Let R be a ring and M a finite R-module. Consider the set of submodules of M defined by $\Lambda = \{IM \mid I \subseteq R \quad \text{an ideal}\}$ and suppose that Λ satisfies the a.c.c. and the d.c.c. Show that M is an Artinian R-module.