

Homework 1 (due: 10-7-11).

All rings are commutative with identity!

- (1) [4pts] Let R be a finite ring. Show that $R^* = \text{NZD}(R)$.
- (2) [8pts] Suppose that R is a subring of a ring S and that R is a direct summand of S as an R -module (i.e. $S = R \oplus N$ as an R -module).
- (a) Prove that each ideal is the contraction of an ideal of S .
- (b) Conclude that R is Noetherian (Artinian) if S is Noetherian (Artinian).
- (3) [16pts] Let R be a ring.
- (a) Suppose that $I \subseteq R$ is an ideal which is not principal, but every ideal which properly contains I is principal. Show that I is a prime ideal.
- (b) If every prime ideal of R is principal then every ideal of R is principal.
- (4) [8pts] Let R be a ring and $S \subseteq R$ a nonempty subset. Show that the following are equivalent:
- (a) $R - S$ is the union of prime ideals.
- (b) $1 \in S$ and $(ab \in S \Leftrightarrow a \in S \text{ and } b \in S)$.
- (5) [10pts] Let R be an integral domain, $P = \{p_\lambda\}_{\lambda \in \Lambda}$ the set of prime elements of R , and S the multiplicative set generated by P and R^* .
- (a) Show that S is saturated.
- (b) Show that the following are equivalent:
- (i) R is a UFD (factorial).
 - (ii) $S^{-1}R$ is the quotient field of R .
 - (iii) Each proper prime ideal of R contains some p_λ .
- (6) [12pts] Assumptions as in problem (5). Suppose in addition that every nonzero nonunit of R is a finite product of irreducible elements of R . Show that if $S^{-1}R$ is a UFD, then R is a UFD.

(7) [8pts] Let M be an R -module and $I \subseteq R$ an ideal. Suppose that $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subseteq R$ with $I \subseteq \mathfrak{m}$. Show that $M = IM$.

(8) [8pts] Suppose that R is an integral domain. An R -module M is called *torsion free* if for all $a \in R - (0)$ and all $m \in M - (0)$ it holds that $am \neq 0$. Show that M is a torsion free R -module if and only if $M_{\mathfrak{m}}$ is a torsion free $R_{\mathfrak{m}}$ -module for all maximal ideals $\mathfrak{m} \subseteq R$.

(9) [16pts] Let R be a ring and $f = \sum_{i=0}^n a_i x^i \in R[x]$ be an element in the polynomial ring over R . Show:

(a) f is invertible in $R[x]$ if and only if $a_0 \in R^*$ and for all $i \geq 1$: a_i is nilpotent.

(b) f is a zerodivisor in $R[x]$ if and only if there is an element $b \in R - (0)$ so that $bf = 0$.

(10) [10pts] For any ring R show that $\text{Jrad}(R[x]) = \text{nil}(R[x])_i$