309 Worksheet 8.2

Let A and C be similar $n \times n$ matrices and P an invertible matrix with $P^{-1}CP = A$. In the following $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{R}^n$.

(1) Show that $B_P = \{P\mathbf{e}_1, \ldots, P\mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

P not only defines the isomorphism $\mu_P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, but we may also consider P as a change-of-basis matrix.

(2) Show that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathrm{id}_{\mathbb{R}^n}} & \mathbb{R}^n \\ {\scriptstyle [\,]_{B_P}} \big\downarrow & & {\scriptstyle [\,]_{B}} \big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n \end{array}$$

commutes. (*Hint:* Test commutativity on the basis vectors of B_P.)

Hence P is the change-of-basis matrix for changing from basis B_P to basis B in $\mathbb{R}^n$.

(3) Show:

(a) $[\,]_B = \mathrm{id}_{\mathbb{R}^n}$

(b) $[\,]_{B_P} = \mu_{P^{-1}}$

(4) Show that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathrm{id}_{\mathbb{R}^n}} & \mathbb{R}^n \\ {[\,]_B}\big\downarrow & & {[\,]_{B_P}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_{P^{-1}}} & \mathbb{R}^n \end{array}$$

commutes.

This gives a big commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}^n & \xrightarrow{\mathrm{id}_{\mathbb{R}^n}} & \mathbb{R}^n & \xrightarrow{\mu_C} & \mathbb{R}^n & \xrightarrow{\mathrm{id}_{\mathbb{R}^n}} & \mathbb{R}^n \\ {[\,]_{B_P}}\big\downarrow & & {[\,]_B}\big\downarrow & & {[\,]_B}\big\downarrow & & {[\,]_{B_P}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n & \xrightarrow{\mu_C} & \mathbb{R}^n & \xrightarrow{\mu_{P^{-1}}} & \mathbb{R}^n \end{array}$$

(5) Show that

(a) All squares in the diagram commute.

(b) The big diagram can be shortened to a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mu_C} & \mathbb{R}^n \\ {[\,]_{B_P}}\big\downarrow & & {[\,]_{B_P}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_{P^{-1}CP}} & \mathbb{R}^n \end{array}$$

(c) $A = P^{-1}CP$ is the matrix of μ_C with respect to the basis B_P.

Summary: Matrices A and C are similar if and only if the corresponding linear operators μ_A and μ_C are the same linear maps up to a base change.