309 Worksheet 6.6

Let $T : V \longrightarrow W$ be a linear transformation of finite-dimensional vector spaces. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ be bases of V and $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, $C' = \{\mathbf{u}'_1, \ldots, \mathbf{u}'_m\}$ bases of W. Suppose that;

(i) A is the matrix of T with respect to bases B and C, that is the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle [\,]_B}\big\downarrow & & {\scriptstyle [\,]_C}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^m \end{array}$$

commutes.

((ii) P is the change-of-basis matrix for changing from basis B' to basis B (of V), that is, the diagram:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{id}_V} & V \\ {\scriptstyle [\,]_{B'}}\big\downarrow & & {\scriptstyle [\,]_B}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n \end{array}$$

commutes.

(c) Q is the change-of-basis matrix for changing from basis C to basis c' (of W), that is the diagram

$$\begin{array}{ccc} W & \xrightarrow{\mathrm{id}_W} & W \\ {\scriptstyle [\,]_C}\big\downarrow & & {\scriptstyle [\,]_{C'}}\big\downarrow \\ \mathbb{R}^m & \xrightarrow{\mu_Q} & \mathbb{R}^m \end{array}$$

commutes.

This yields a big diagram:

$$\begin{array}{ccccccc} V & \xrightarrow{\mathrm{id}_V} & V & \xrightarrow{T} & W & \xrightarrow{\mathrm{id}_W} & W \\ {\scriptstyle [\,]_{B'}}\big\downarrow & & {\scriptstyle [\,]_B}\big\downarrow & & {\scriptstyle [\,]_C}\big\downarrow & & {\scriptstyle [\,]_{C'}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^m & \xrightarrow{\mu_Q} & \mathbb{R}^m \end{array}$$

Show:

(a) All squares in the big diagram commute.

(b) The big diagram can be shortened to a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle [\,]_{B'}}\big\downarrow & & {\scriptstyle [\,]_{C'}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_{QAP}} & \mathbb{R}^m \end{array}$$

(c) QAP is the matrix of T with respect to bases B' and C'.