**309 Worksheet 6.5**

(1) Let $V$ be a finite-dimensional vector space with ordered bases $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. The identity map $\mathrm{id}_V : V \longrightarrow V$ is a linear transformation which yields a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{id}_V} & V \\ {\scriptstyle [\,]_B}\big\downarrow & & {\scriptstyle [\,]_{B'}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n \end{array}$$

where $P$ is the change-of-basis matrix from basis $B$ to basis $B'$. Show:

(a) $\mu_P$ is one-to-one and onto.
(b) $P$ is an invertible matrix.
(c) $P^{-1}$ is the change-of-basis matrix for changing from basis $B'$ to $B$.

(2) Let $V$ be a finite-dimensional vector space with ordered bases $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $B' = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, and $B'' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$. The identity map yields a commutative diagram:

$$\begin{array}{ccccc} V & \xrightarrow{\mathrm{id}_V} & V & \xrightarrow{\mathrm{id}_V} & V \\ {\scriptstyle [\,]_B}\big\downarrow & & {\scriptstyle [\,]_{B'}}\big\downarrow & & {\scriptstyle [\,]_{B''}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_P} & \mathbb{R}^n & \xrightarrow{\mu_{P'}} & \mathbb{R}^n \end{array}$$

where $P$ is the change-of-basis matrix from $B$ to $B'$ and $P'$ is the change-of-basis matrix from $P'$ to $P''$. Show:

(a) All three squares are commutative.

(b) The above diagram can be shortened to a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{id}_V} & V \\ {\scriptstyle [\,]_B}\big\downarrow & & {\scriptstyle [\,]_{B''}}\big\downarrow \\ \mathbb{R}^n & \xrightarrow{\mu_{P'P}} & \mathbb{R}^n \end{array}$$

(c) $P'P$ is the change-of-basis matrix from basis $B$ to basis $B''$.