309 Worksheet 6.4

(1) Let V be a finite dimensional vector space with ordered basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $[\,]_B : V \longrightarrow \mathbb{R}^n$ denote the coordinate function of V with respect to B, that is, if $\mathbf{v} \in V$ with $\mathbf{v} = a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$ then $[\mathbf{v}]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and let $L_B : \mathbb{R}^n \longrightarrow V$ denote the function defined by $L_B \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = r_1\mathbf{v}_1 + \ldots r_n\mathbf{v}_n$. Show

(a) $[\,]_B$ and L_B are linear transformations.
(b) $[\,]_B$ and L_B are inverse to each other.

(2) Let V and W be finite-dimensional vector spaces with ordered bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and $B' = B = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of W. Let $T : V \longrightarrow W$ be a linear transformation. Consider the commutative diagram

$$\begin{CD} V @>T>> W \\ @V{[\,]_B}VV @V{[\,]_{B'}}VV \\ \mathbb{R}^n @>\mu_A>> \mathbb{R}^m \end{CD}$$

where A is the matrix of T relative to B and B'. Show

(a) There is exactly one map $\mu_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that the above diagram commutes.
(b) T is one-to-one if and only if μ_A is one-to-one.
(c) T is onto if and only if μ_A is onto.
(d) T has an inverse function if and only if the matrix A is invertible.