309 Worksheet 6.3

Theorem (1). *Let V and W be vector spaces and $T : V \longrightarrow W$ a linear transformation. Assume that V is finite-dimensional with basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Show:*

(a) *T is one-to-one if and only if the set $T(B) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is linearly independent.*

(b) *T is onto if and only if the set $T(B) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ spans W.*

(c) *T is onto and one-to-one if and only if the set $T(B) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis of W.*

(2) Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow \\ S & \xrightarrow{\ell} & T \end{array}$$

be a commutative diagram. Suppose that the maps h and g are onto and one-to-one. Show:

(a) f is one-to-one if and only if ℓ is one-to-one.
(b) f is onto if and only if ℓ is onto.
(c) f is one-to-one and onto if and only if ℓ is one-to-one and onto.

(3) Let X, Y, S, and T be sets and $h : X \longrightarrow S$ and $g : Y \longrightarrow T$ one-to-one and onto maps.

(a) Let $f : X \longrightarrow Y$ be a map. Show that there is a unique map $\ell : S \longrightarrow T$ so that the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ S & \xrightarrow{\quad \ell \quad} & T \end{array}$$

commutes.

(b) Let $\ell : S \longrightarrow T$ be a map. Show that there is a unique map $f : X \longrightarrow Y$ so that the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ S & \xrightarrow{\quad \ell \quad} & T \end{array}$$

commutes.