309 Worksheet 6.1

(1) Let $f : X \longrightarrow Y$ be a function. Negate the notion of 'one-to-one' and 'onto' functions using 'for all' and 'exists':

(a) f is not one-to-one if

(b) f is not onto if

(2) Let $X = \{1, 2, 3, 4\}$. Describe a set X and a function $f : X \longrightarrow Y$ so that f is: (Note that in all 4 cases the set Y may vary.)

(a) onto Y but not one-to-one

(b) one-to-one but not onto

(c) both one-to-one and onto

(d) neither one-to-one nor onto

(3) Let $f : X \longrightarrow Y$ be a function. Show that f is one-to-one and onto if and only if there is a function $g : Y \longrightarrow X$ so that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

(4) A diagram of functions

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ S & \xrightarrow{\ell} & T \end{array}$$

is called *commutative* if $gf = \ell h$, that is, the two composition functions $gf, \ell h : X \longrightarrow T$ are identical.

Verify that the following diagrams are commutative:

(a)

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ \mathbb{R} & \xrightarrow{\ell} & \mathbb{R} \end{array}$$

where $f(x) = 2x, g(x) = \frac{1}{2}x, h(x) = x + 2$, and $\ell(x) = x - 2$.

(b)

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^3 \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ \mathbb{R}^4 & \xrightarrow{\ell} & \mathbb{R}^3 \end{array}$$

where $f(x, y) = (2x, 3y, 4), g(x, y, z) = (x, y, 0), h(x, y) = (2x, y, 0)$, and $\ell(x, y, z, u) = (x, 3y, 0)$.