309 Worksheet 5.3 (section 5.2)

Let $A \in \mathbb{M}(n,n)$. Suppose we want to solve the n linear systems:

$$(*) \qquad \begin{aligned} A\mathbf{x}_1 &= \mathbf{e}_1 \\ A\mathbf{x}_2 &= \mathbf{e}_2 \\ &\vdots \\ A\mathbf{x}_n &= \mathbf{e}_n \end{aligned}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{R}^n$. We can do this by considering n augmented $n \times (n+1)$ matrices $[A \,|\, \mathbf{e}_1], [A \,|\, \mathbf{e}_2], \ldots, [A \,|\, \mathbf{e}_n]$. In each case, we bring A into reduced echelon form D by a sequence of elementary row operations and perform the same sequence of elementary row operations on the $(n+1)$st column $\mathbf{e}_i$. Note that the same sequence of elementary row operations brings A into reduced echelon in ALL n cases. Hence we can shorten this process by combining the n augmented matrices $[A \,|\, \mathbf{e}_i]$ into one augmented $n \times 2n$ matrix $[A \,|\, I_n]$. Then we perform the same sequence of row operations on A and I_n to bring A into reduced echelon form D. The result is an $n \times 2n$ matrix $[D \,|\, C]$ with D the reduced echelon form of A and C a matrix obtained from I_n by the same sequence of row operations that has been applied to A. Remember that Problem (4) (b) of worksheet 5.2 implies that no row of C consists entirely of zeros!

Problem (*1*) Let $A \in \mathbb{M}(n,n)$. Show that there is an $n \times n$ matrix C with $AC = I_n$ if and only if all n linear systems $(*)$ are solvable. If $AC = I_n$ what are the columns of C?

Problem (2) Let $A \in \mathbb{M}(n,n)$. Show:

(a) If the reduced echelon form of A contains a row which consists entirely of zeros, then for all matrices $B \in \mathbb{M}(n,n)$, $AB \neq I_n$.

(b) If the reduced echelon form D of A equals I_n, then there is a matrix $C \in \mathbb{M}(n,n)$ with $AC = I_n$. Moreover, such a matrix C can be obtained by a sequence of elementary row operations which reduces the augmented matrix $[A \mid I_n]$ to $[I_n \mid C]$.

We just have shown the following theorem:

Theorem 1. *Let $A \in \mathbb{M}(n,n)$.*

(a) *If the reduced echelon form of A contains a row which consists entirely of zeros, then for all matrices $B \in \mathbb{M}(n,n)$, $AB \neq I_n$.*
(b) *If the reduced echelon form of A is I_n, then the augmented matrix $[A \mid I_n]$ can be reduced to a matrix $[I_n \mid C]$ where $AC = I_n$.*

This method provides even more. We claim:

Theorem 2. *Let $A \in \mathbb{M}(n,n)$ and suppose that the augmented matrix $[A \mid I_n]$ can be reduced to $[I_n \mid C]$ by a sequence of elementary row operations. Then $AC = CA = I_n$.*

Proof. We have already shown that $AC = I_n$. By assumption the matrix $[A \,|\, I_n]$ can be reduced to $[I_n \,|\, C]$ by elementary row operations. This implies that I_n is the reduced echelon form of A. Performing an elementary row operation on a matrix B is the same as multiplying B from the left by the corresponding elementary matrix. Thus there are elementary matrices $E_1, E_2, \ldots, E_m$ so that

$$E_m E_{m-1} \ldots E_2 E_1 A = I_n.$$

In the augmented matrix $[A \,|\, I_n]$ the same sequence of elementary row operations has been applied to I_n in order to obtain the matrix C. This means:

$$E_m E_{m-1} \ldots E_2 E_1 I_n = C = E_m E_{m-1} \ldots E_2 E_1$$

and showing that also $CA = I_n$.

Definition. Let $A \in \mathbb{M}(n,n)$. A is called *invertible* or *nonsingular* if there is a matrix $C \in \mathbb{M}(n,n)$ with $AC = CA = I_n$. A matrix with this property is called a *multiplicative inverse* of A.

Theorem 3. *An invertible matrix has a unique inverse, written A^{-1}.*

Proof. Suppose that $C, C' \in \mathbb{M}(n,n)$ with $AC = CA = I$ and $AC' = C'A = I$. Multiply the equation

$$AC = I$$

by C' from the left. Thus:

$$C'(AC) = C'I = C'.$$

By the associative law for multiplication of matrices:

$$C'(AC) = (C'A)C$$

and by assumption $C'A = I$. Thus

$$C'(AC) = (C'A)C = IC = C = C'.$$

Theorem 4. *If a matrix is invertible, then A^{-1} is also invertible. In this case $(A^{-1})^{-1} = A$.*

Proof. Obviously,

$$A^{-1}A = AA^{-1} = I$$

and by the uniqueness of the inverse $(A^{-1})^{-1} = A$.

Theorem 5. *Suppose $A, B \in \mathbb{M}(n,n)$. If A and B are invertible, then so is AB. In this case, $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. Verify that

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= I \quad \text{and} \\ (B^{-1}A^{-1})(AB) &= I \end{aligned}$$

Corollary 6. *If $A, B \in \mathbb{M}(n,n)$ with $AB = I$, then $BA = I$.*

Proof. For all $1 \leq j \leq n$ the columns B_j of B are solutions to the linear system $A\mathbf{x} = \mathbf{e}_j$. By problem (2)(b) the augmented matrix $[A \,|\, I_n]$ reduces by elementary row operations to $[I_n \,|\, C]$ where $AC = CA = I$. Hence $C = CI = C(AB) = (CA)B = IB = B$ and also $BA = I$ by Theorem (2).

Corollary 7. *If $A, B \in \mathbb{M}(n,n)$ with $BA = I$, then $AB = I$.*

Proof. Use Corollary 6 and interchange the role of A and B.

Definition. The *rank* of an $m \times n$ matrix A, denoted rank$A$, is the number of leading ones in the reduced echelon form of A.

Theorem 8. *An $n \times n$ matrix A has an inverse C if and only if rank$A = n$.*

Proof. Here comes your proof:

Note that the rank of an $m \times n$ matrix A equals the dimension of the row space $R(A)$ of A. Of course one can similarly define the column space $C(A)$ of A as the subspace of $\mathbb{R}^m$ which is spanned by the columns $A_1, \ldots, A_n$ of A. Then the following equality holds true:

$$\dim R(A) = \dim C(A) = \text{rank} A.$$

(without proof)