309 Worksheet 5.2

An $n \times n$ *elementary matrix* is a matrix obtained from the $n \times n$ identity matrix I_n by one elementary operation.

$E_{i \leftrightarrow j}$ is obtained from I_n by interchanging the ith and the jth row.
E_{ai} is obtained from I_n by multiplying the ith row by the nonzero constant a.
E_{ai+j} is obtained from I_n by adding a times the ith row to the jth row.

(1) Let A be an $n \times m$ matrix and E an $n \times n$ elementary matrix. Show:

(a) $E_{i \leftrightarrow j}A$ is the matrix obtained from A by interchanging the ith and the jth row of A.

(b) $E_{ai}A$ is the matrix obtained from A by multiplying the ith row of A by $a \neq 0$.

(c) $E_{ai+j}A$ is the matrix obtained from A by adding a times the ith row to the jth row of A.

(2) Let B be an $m \times n$ matrix and E an $n \times n$ elementary matrix. Show:

(a) $BE_{i \leftrightarrow j}$ is the matrix obtained from B by interchanging the ith and the jth column of B.

(b) BE_{ai} is the matrix obtained from B by multiplying the ith column of B by a.

(c) BE_{ai+j} is the matrix obtained from B by adding a times the ith column to the jth column of B.

(3) Show:

(a) $E^2_{i \leftrightarrow j} = I_n$

(b) $E_{ai}E_{a^{-1}i} = E_{a^{-1}i}E_{ai} = I_n$

(c) $E_{ai+j}E_{(-a)i+j} = E_{(-a)i+j}E_{ai+j} = I_n$

Let $A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ be an $m \times n$ matrix where $\mathbf{a}_i \in \mathbb{M}(1, n)$ denote the rows of A. The *row space* of A is defined to be the subspace $R(A) = \text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \subseteq \mathbb{M}(1, n)$.

(4) Show:
(a) If A' is a matrix obtained from A by a sequence of elementary row operations, then $R(A') = R(A)$. In particular, $\dim(R(A')) = \dim(R(A))$.

(b) Let I_n be the $n \times n$ identity matrix and C a matrix obtained from I_n by a sequence of elementary row operations. Then no row of C consists entirely of zeros.