Constructions of 4-Manifolds

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Michigan State University
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Joint work with Ron Stern

Things which are seen are temporal,
but the things which are not seen are eternal.
B. Stewart and P.G. Tait

4-Manifold basic facts
Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H_2(X;\mathbb{Z}) \otimes H_2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (PD(\alpha) \cup PD(\beta))[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic $\iff$ they have the same e, sign, and type.

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Smooth structures

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

Goal of this lecture — Discuss techniques used to study this conjecture

Seiberg-Witten Invariants

$$\mathrm{SW}_X : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$

- $\mathrm{SW}(k) \neq 0$ for only finitely many k: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot k|$$

for every basic class k. (Adjunction Inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

- $\mathrm{SW}_X(\kappa) = \pm 1$, $\kappa = c_1$(symplectic manifold with $b^+ > 1$) [Taubes].
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Geography

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$c = 3\text{sign} + 2e$
$\chi = \frac{\text{sign}+e}{4} = \frac{b^+ + 1}{2}$

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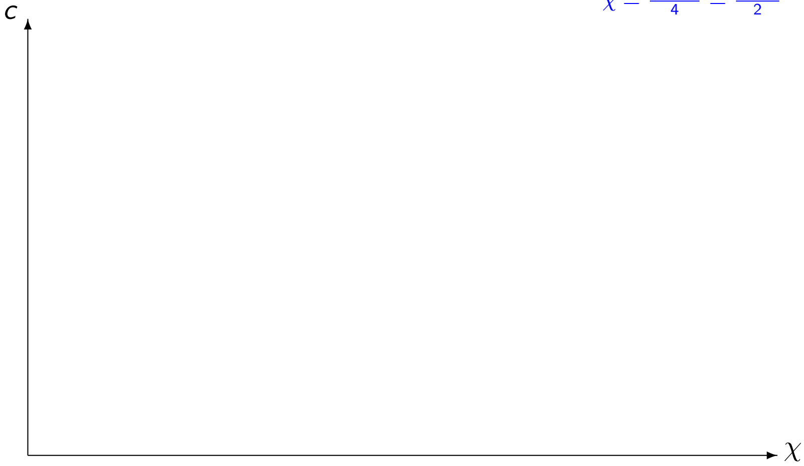

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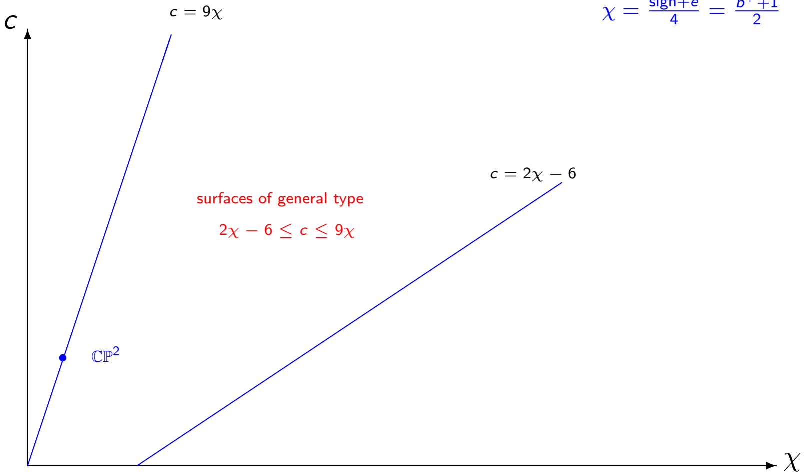

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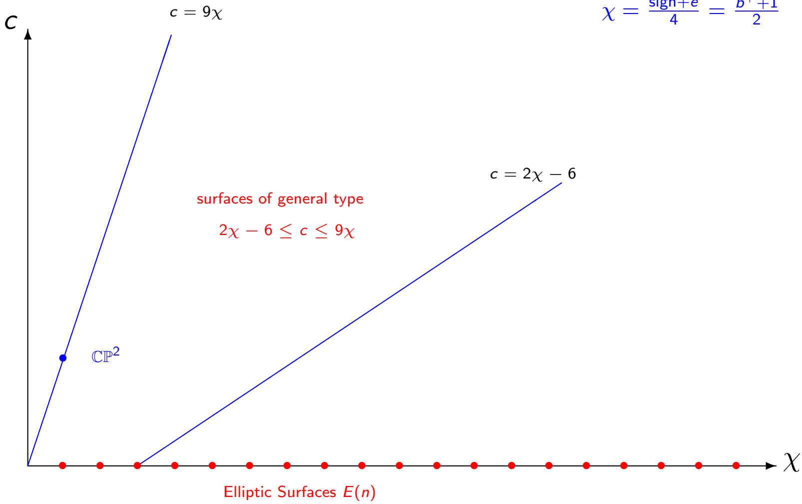

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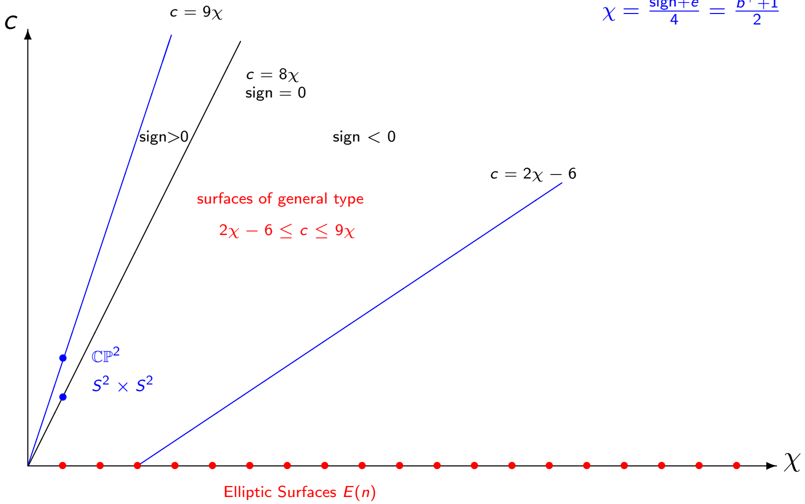

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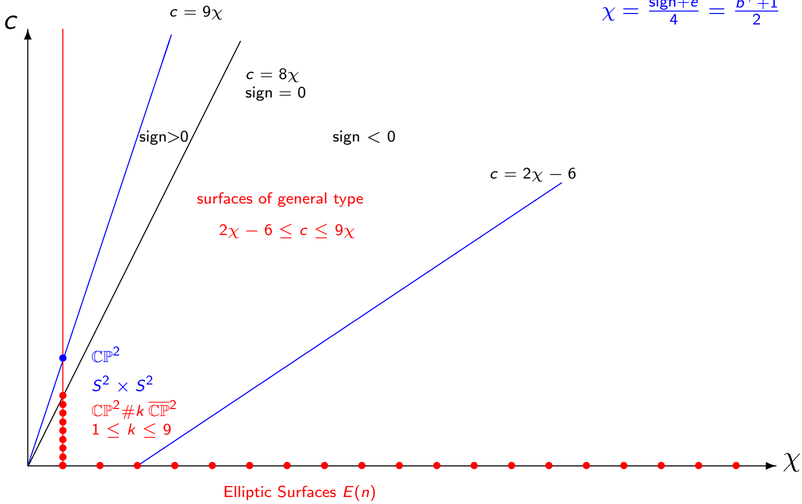

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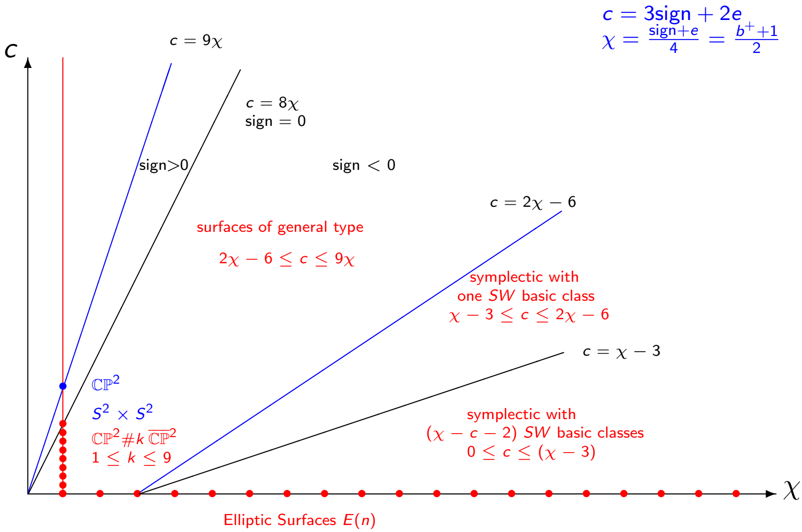

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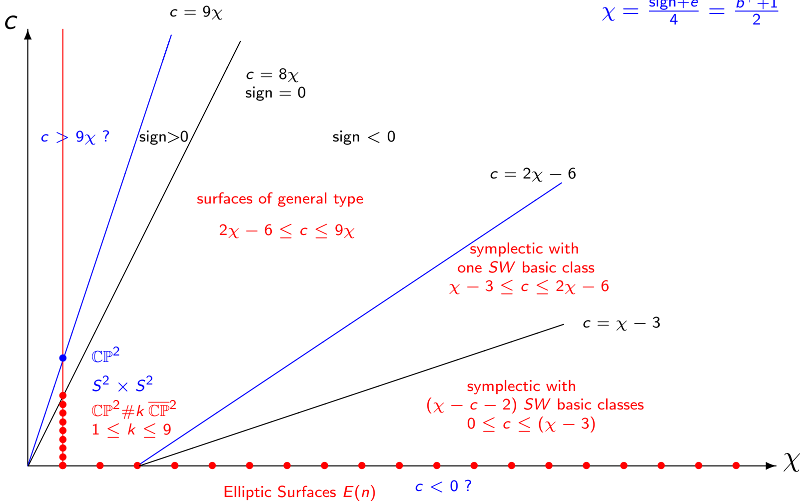

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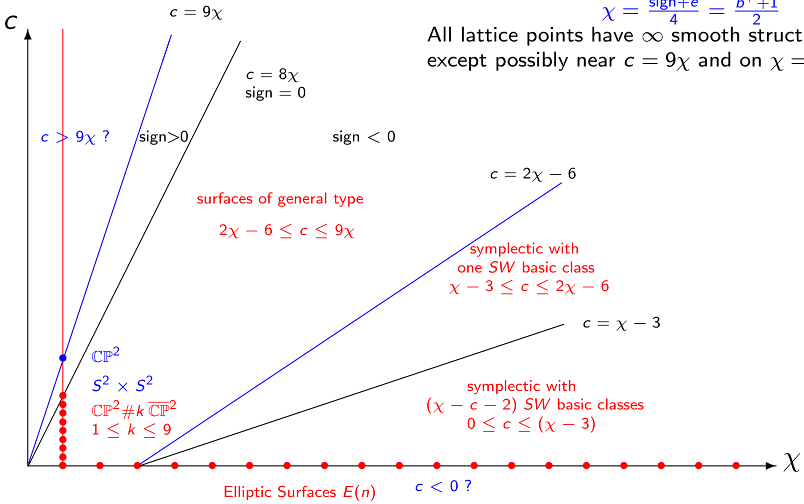
All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi = 1$



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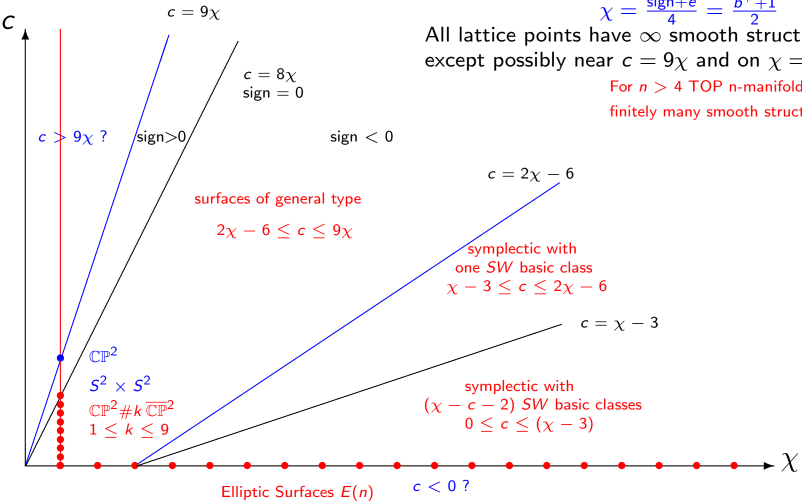
For $n > 4$ TOP n-manifolds have finitely many smooth structures



Nullhomologous Tori

- One way to try to prove the conjecture — Find a "dial" to change the smooth structure at will.
- This dial: Surgery on nullhomologous tori

T: any self-intersection 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$$\varphi(\mathrm{pt} \times \partial D^2) = \text{surgery curve}$$

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T)) = \mathbb{Z}^3$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

This operation does not change $e(X)$ or $\mathrm{sign}(X)$

Note: $X_T(0, 0, 1) = X$

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Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T)) = \mathbb{Z}^3$

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The Morgan, Mrowka, Szabo Formula

Describes how surgery on a torus changes the Seiberg-Witten invariant

T: torus in X with self-intersection $= 0$ Nbd $= S^1 \times S^1 \times D^2$
Do (p, q, r) - surgery to get $X_T(p, q, r)$

Roughly

$$\mathcal{SW}_{X_T(p,q,r)} = p\,\mathcal{SW}_{X_T(1,0,0)} + q\,\mathcal{SW}_{X_T(0,1,0)} + r\,\mathcal{SW}_{X_T(0,0,1)}$$

Example: $S^1 \times \frac{p}{q}$-Dehn surgery on circle C in 3-manifold Y
Corresponds to $(0, q, p)$-surgery on the torus $T = S^1 \times C \subset X = S^1 \times Y$ to get X'

$$\mathcal{SW}_{X'} = p\,\mathcal{SW}_X + q\,\mathcal{SW}_{X_0}$$

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First Application: Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

- $$X_K = X \smallsetminus N_T \cup S^1 \times (S^3 \smallsetminus N_K)$$

Note: $S^1 \times (S^3 \smallsetminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K
  (So X_K homeo to X)
- If K is fibered and X and T both symplectic; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusions

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.
- X, T symplectic, K fibered $\Rightarrow X_K$ symplectic. So there is an infinite family of distinct symplectic manifolds homeo X.

e.g. $X = K3$, $\mathcal{SW}_X = 1$, $\mathcal{SW}_{X_K} = \Delta_K(t^2)$

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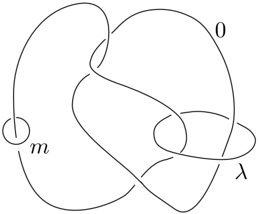
Knot surgery and nullhomologous tori

Knot surgery on torus T in 4-manifold X with knot K:

$X_K = X \#_{T = S^1 \times m} S^1 \times$



$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings

- Weakness of construction: Need T to be homologically essential
- Open conjecture: If $\chi(X)(= \frac{e(X)+\text{sign}(X)}{4}) > 1$, then X contains a homologically essential minimal genus torus T with trivial normal bundle (in the complement of all the basic classes)
- If X homeomorphic to CP^2 blown up at 8 or fewer points, then X contains no such torus - so what can we do there?

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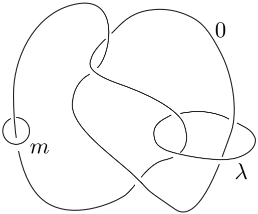

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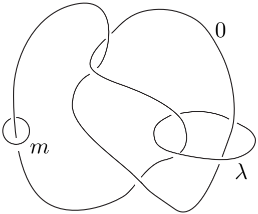

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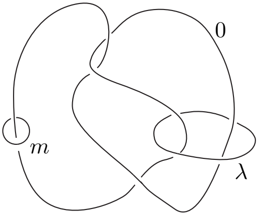
$X_K = X \#_{T = S^1 \times m} \; S^1 \times$ 

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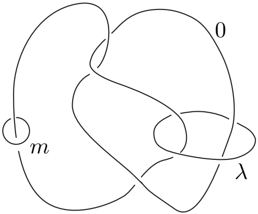
Knot surgery and nullhomologous tori

Knot surgery on torus T in 4-manifold X with knot K:

$X_K = X \#_{T = S^1 \times m} S^1 \times$ 

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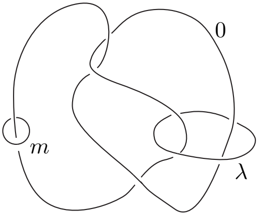
$X_K = X \#_{T = S^1 \times m} S^1 \times$ 

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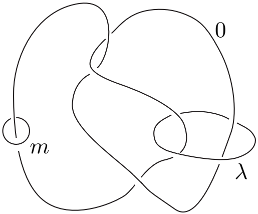

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$$E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$$

Elliptic surface F: fiber (torus of square 0) $N_F = S^1 \times S^1 \times D^2$

$F = S^1 \times f, \;\; \Lambda = S^1 \times \lambda$

Λ: Nullhomologous torus in $E(1)$
Whitehead double of fiber

s lies in a section

What is the result of surgery on Λ?

$X(1/n) = S^1 \times$ ($\frac{1}{n}$-surgery on λ) homeo to $E(1)$

$$\mathcal{SW}_{X(1/n)} = \mathcal{SW}_{E(1)} + n\mathcal{SW}_{X_0}$$
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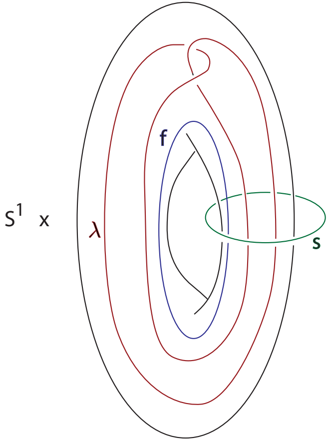

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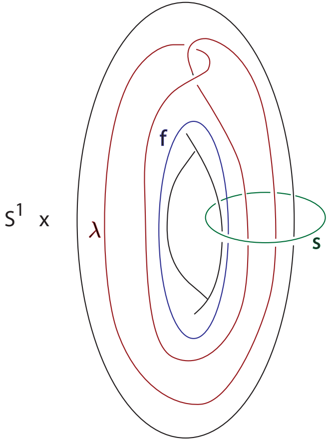

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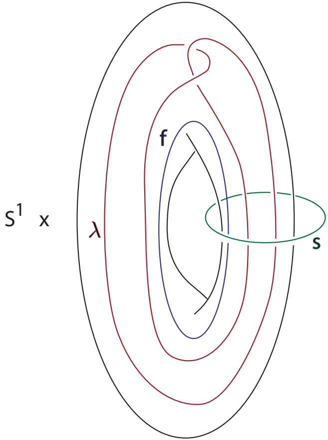

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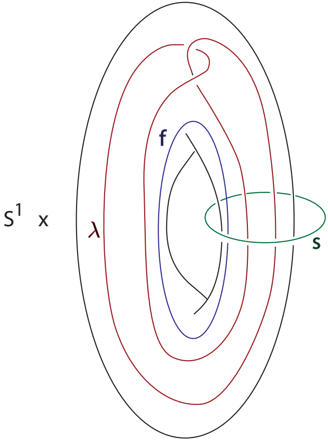

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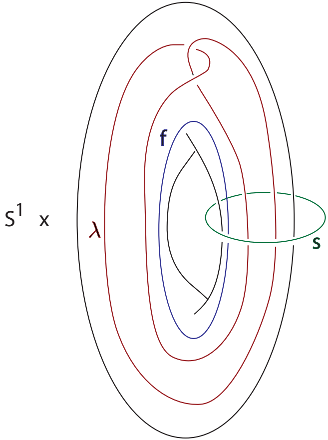

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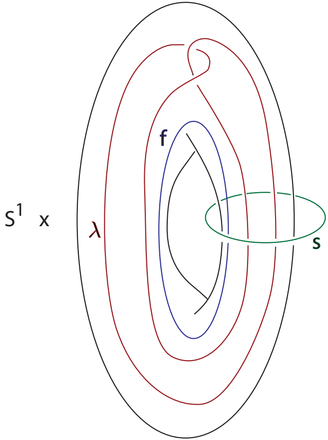

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A Surgery Duality

T: self-intersection 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$

Basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ $\{\alpha, \beta\}$: pushoffs of basis for $H_1(T)$

Compare two situations:

(a) T primitive, pushoff curve $\beta \subset N_T$ essential in $X \setminus T$

Do $S^1 \times p/1$ - surgery on T (i.e. (0,1,p)-surgery)

$\Rightarrow$ $T_{p/1}$ nullhomologous in $X_T(p/1)$.

(Its meridian is $\beta + p\mu_T \sim \beta \not\sim 0$ in $X \setminus N_T$.)

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β' bounds in $X_T(p/1) \setminus N_{T_{p/1}} = X \setminus N_T$.

(b) T nullhomologous, β bounds in $X \setminus N_T$

$S^1 \times 0/1$ (*i.e.* nullhomologous) surgery on T gives (a).

(a) $\longrightarrow$ (b) reduces b_1 by 1 and increases H_2 by a hyperbolic pair.

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Reverse Engineering

- Difficult to find useful nullhomologous tori as in applications above

  Recall: $\mathcal{SW}_{X_{T(p/1)}} = \mathcal{SW}_X + p\,\mathcal{SW}_{X_T(0/1)}$

  IDEA: First construct $X_T(0/1)$ so that $\mathcal{SW}_{X_T(0/1)} \neq 0$ and then surger to reduce b_1.

- Procedure to insure the existence of effective nullhomologous tori

1. Find model manifold M with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $\mathcal{SW} \neq 0$.
2. Find b_1 disjoint essential tori in M containing generators of H_1. Surger to get manifold X with $H_1 = 0$. Want result of each surgery to have $\mathcal{SW} \neq 0$ (except perhaps the very last).
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Luttinger Surgery

M: symplectic manifold T: Lagrangian torus in M

Preferred framing for T: Lagrangian framing
w.r.t. which all pushoffs of T remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $\beta =$ Lagrangian pushoff,
$M_T(\pm 1) = (0, 1, \pm 1)$-surgery is a symplectic mfd

$\Longrightarrow$ if $b^+ > 1$, $M_{T,\beta}(\pm 1)$ has $\mathcal{SW} \neq 0$

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Families

- **The $\mathcal{SW}$ condition**
  If M is symplectic and surgery tori are Lagrangian and we do (± 1)-surgeries with respect to the Lagrangian framings, each resultant manifold will be symplectic and have $\mathcal{SW} \neq 0$.
- **Simple connectivity**
  Easier in some cases than others
- **Infinite families**

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff and $\mathcal{SW}_{X_{\Lambda,\lambda}(1/n)}$ all different

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$\Longrightarrow$ Infinite family

Model Manifolds for $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$

Basic Pieces: $X_0, X_1, X_2, X_3 X_4 X_5$

X_0: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0,1)$
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Model Manifolds

Basic Pieces: X_3

$X_3 = S^2 \times T^2 \# 3\,\overline{\mathbb{CP}}^2, \quad c_1^2(X_3) = -3,\ \chi(X_3) = 0$

In $S^2 \times T^2$ there is an embedded torus T' representing $2T^2$.
Consider configuration $T' + T^2 + S^2$ which has 3 double points.
Blowup one double point on T' and smooth the other two double points. Then blow up at two more points on the result.
Get Σ: genus 2, square 0 homologous to $3T^2 + S^2 - 2E_1 - E_2 - E_3$.

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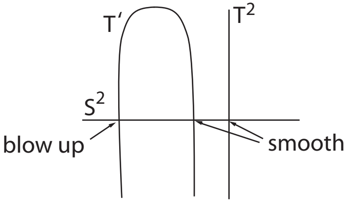


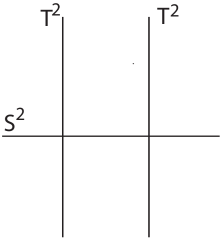
Model Manifolds

Basic Pieces: X_4

$$X_4 = S^2 \times T^2 \# 4\,\overline{\mathbb{CP}}^2, \quad c_1^2(X_4) = -4,\ \chi(X_4) = 0$$

In $S^2 \times T^2$ consider configuration with 2 disjoint copies of T^2 and one S^2. Smooth the double points and then blow up at 4 points to get Σ homologous to $2T^2 + S^2 - E_1 - E_2 - E_3 - E_4$.

Σ has genus 2 and square 0.



Example: Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

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Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has disjoint Lagrangian tori carrying basis for H_1.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. (Li-Liu)
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)
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- Examples first obtained by Baldridge-Kirk and Akhmedov-Park.

Example: Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold:
$X_2 \#_{\Sigma_2} X_0 = ((T^4 \# \mathbb{CP}^2) \# \mathbb{CP}^2) \#_{\Sigma_2} (T^2 \times \Sigma_2)$
$= (Sym^2(\Sigma_2) \# \mathbb{CP}^2) \#_{\Sigma_2} (T^2 \times \Sigma_2) \cong Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has disjoint Lagrangian tori carrying basis for H_1.

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Complex fake projective plane is a complex surface X with $H_*(X;\mathbb{Q}) = H_*(\mathbb{CP}^2;\mathbb{Q})$ but not diffeo to $\mathbb{CP}^2$.

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Finitely many, each has nontrivial $H_1(X;\mathbb{Z})$
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Example: Smooth Fake Projective Planes

Start with elliptic fibration on $E(1)$ with 4 I_3 fibers.

$I_3 \leftrightarrow$ 3 nodal fibers with parallel vanishing cycles

Do knot surgery on $E(1)$ with $K =$ trefoil knot
section becomes torus of self-intersection -1 (Pseudosection)

Red curve isotopic to green and blue curves

Meridian to knot bounds vanishing disk in $E(1) \setminus N_F$

Get disjoint disks of self-intersection -1

Use to surger pseudosection to sphere of square -3

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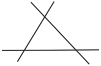

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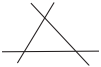

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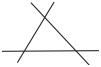

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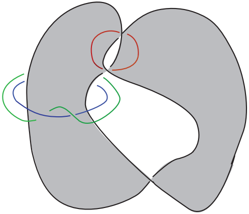

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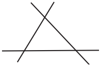

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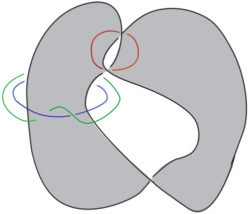

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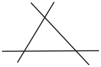

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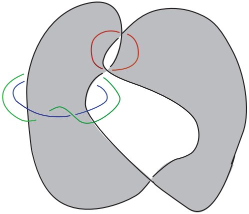

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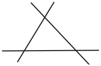

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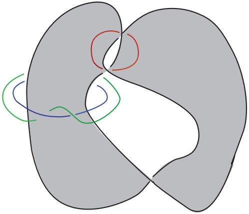

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Smooth Fake Projective Planes

In $E(1)_K$ can arrange

Follow idea of Keum: Collapse three $(-2)-(-2)$ to $c(L(3,-2))$

Take 3-fold branched cover — get homotopy $E(1)$ (nonsingular) $:Y$

Y contains three copies of $(-3)-(-2)-(-2)$.

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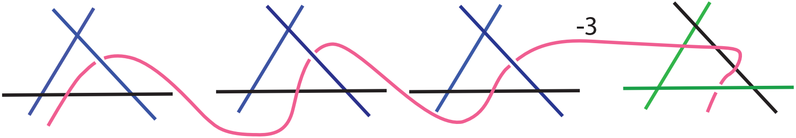

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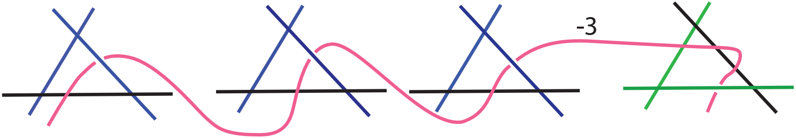

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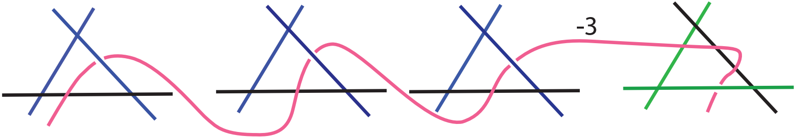

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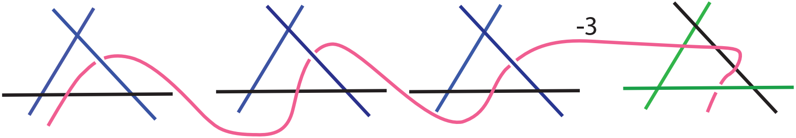

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Families of Smooth Fake Projective Planes

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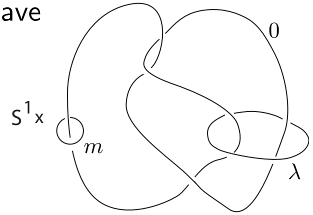

Constructions above can be shown to be disjoint from $S^1 \times \lambda$

p/q-surgeries give $\mathbb{Q}$-homology $E(1)$'s with different SW-invariants

Construction gives $\mathbb{Q}$-homology $\mathbb{CP}^2$'s. SW = ?

They have $Z/7$-actions with different orbit spaces.

Are they irreducible?

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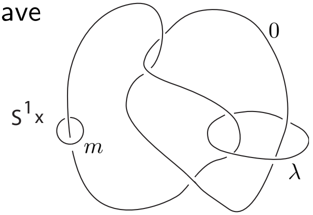

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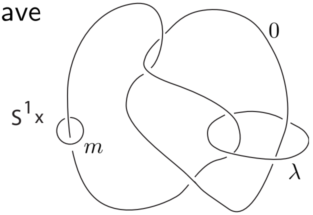

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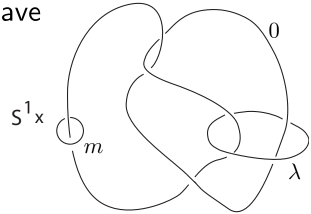

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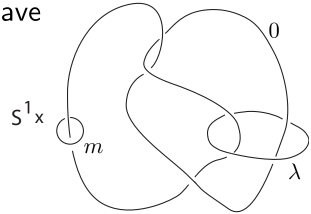

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