Surgery on Nullhomologous Tori

Ron Fintushel

- joint with Ron Stern

CONJ. X: topological 4-mfd with KS-inv't $= 0$

$\Rightarrow X$ admits ∞'ly many distinct smooth str's.

Best way to prove conj.

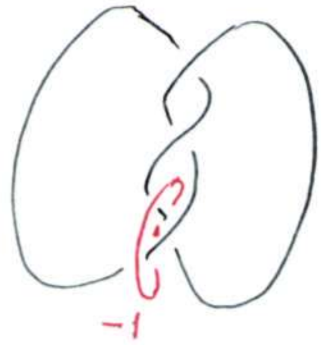
Get dials to rotate

One known way to change smooth str

Knot surgery

Dial = essential torus of self-$\cap = 0$ with $\pi_1 = 0$ complement

Relation with nullhomologous tori



Changes crossing

Macareña

Th'm of Morgan - Mrowka - Szabo

Essentially For $S^1 \times {}^p/_q$ - Dehn surgery on torus T of self - $\cap = 0$, $T = S^1 \times \gamma$

(eg. assume γ is nullhomologous & use nullhomologous framing)

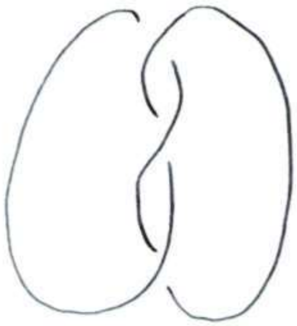
then

$$SW_{X_{p/q}} = p\,SW_X + q\,SW_{X_0}$$

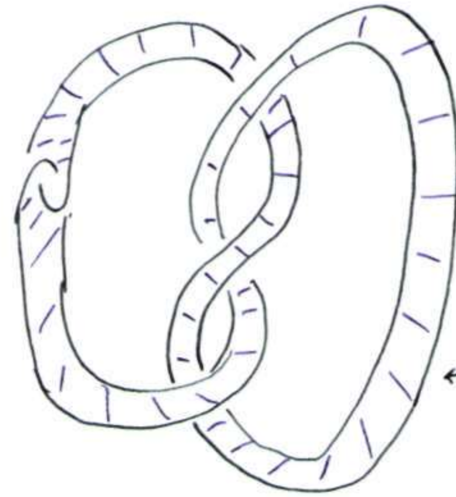
surgery wrt nullhomol. framing

Simple method for producing nullhomologous loops

Whitehead doubling



Knot in some 3-mfd



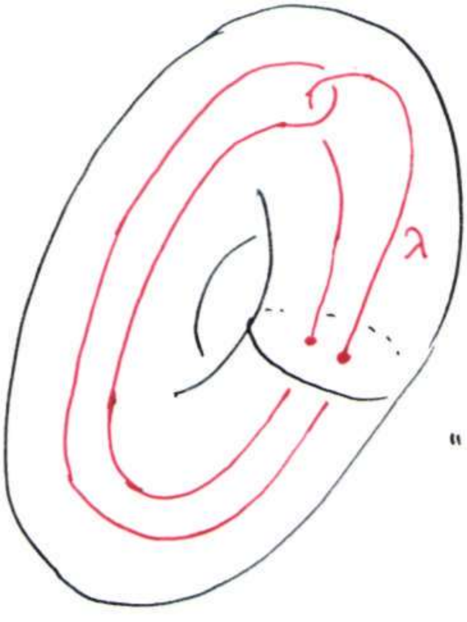
perhaps some twists

A Whitehead double

- can see genus 1 nullhomology

T: essential self-$\cap = 0$ torus in X

$\mathrm{Nbd}\, T = S^1 \times$



$\Lambda = S^1 \times \lambda$

nullhomologous

"Whitehead double of T"

Relate to problem of constructing exotic smooth str's on $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$

History

$k=9$ ∞'ly many smooth str's - Donaldson

$k>9$ Blowup formula

$k=8$ Barlow surface - Kotschick

$k=7$ Park

$k=6$ Stipsicz - Szabo

F - Stern : ∞'ly many smooth str's $k=6,7,8$

PSS $k=5$

Same features for all latter constructions (No min. genus essential tori of square 0)

- Find config of 2-spheres that can be rationally blown down - after first blowing up (many times, perhaps)

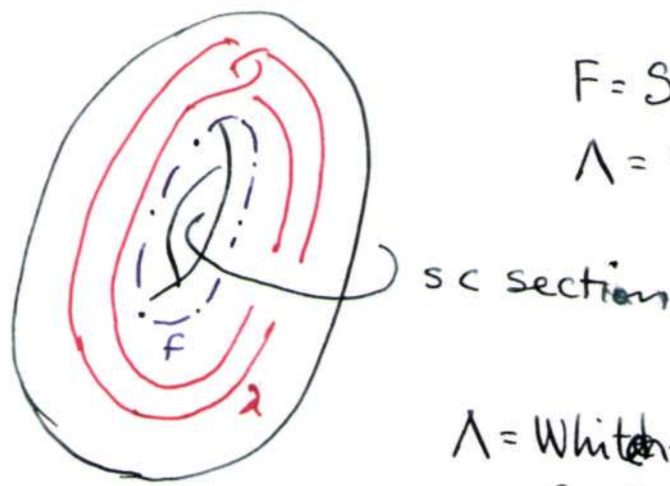
eg $\overset{-9}{\bullet}\!-\!\overset{-2}{\bullet}\!-\!\overset{-2}{\bullet}\!-\!\overset{-2}{\bullet}\!-\!\overset{-2}{\bullet}\!-\!\overset{-2}{\bullet}$ $\partial = L(49,-6)$

Goal - See these examples by 'turning a dial' (rather than using aux. mfds) - Joint work with Ron Stern

$E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$ Elliptic surface

$T^2 = F =$ elliptic fiber (smooth) nbd $N_F \cong S^1 \times (S^1 \times D^2)$

$= S^1 \times$



$F = S^1 \times f$

$\Lambda = S^1 \times \lambda$

$\Lambda =$ Whitehead double of fiber F

What is result of $\frac{1}{n}$ surgery on Λ ?

$E(1)$ admits metric of +'ve scalar curv.

$\Rightarrow SW_{E(1)} = 0$

M-M-Sz Formula $\Longrightarrow$

$$SW_{E(1)_{\Lambda,\frac{1}{n}}} = n\, SW_{E(1)_{\Lambda,0}}$$

$E(1)_{\Lambda,0}$ = mfd obtained by killing longitude of λ via surgery

Has $b_1 = 1$ and $H_2 = H_2(E(1)) \oplus \{\text{hyperbolic pair}\}$

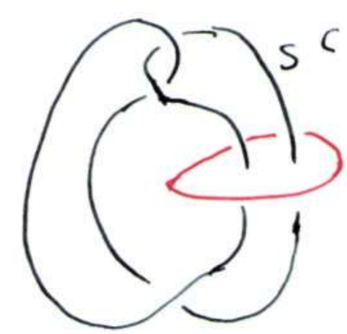
core torus of surgery

+

dual torus = $\left(\begin{array}{l} D^2 \text{ bdd by long. of } \lambda \\ \cup \;\; \text{punctured torus} \end{array}\right)$

Whitehead link symmetric (+ isotopy) $\Rightarrow$

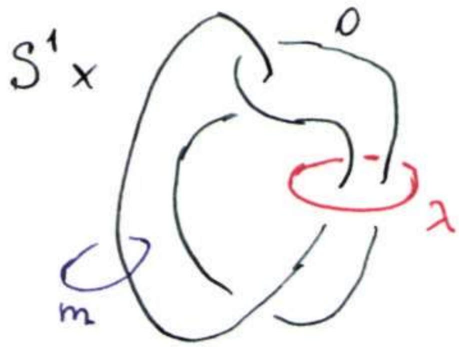
$S^1 \times$ 

$\cong$ $S^1 \times$ 

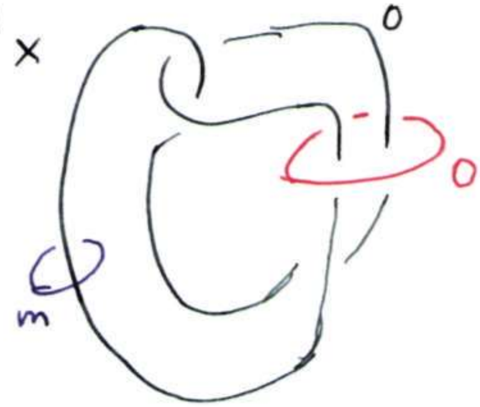
Can achieve this situation directly via knot surgery with $K = s$ (= unknot)

since $E(1)_{\text{unknot}} = E(1)$

$$E(1) = E(1)_{K=s} = E(1) \#_{F=S^1 \times m} S^1 \times$$



$$E(1)_{\Lambda,0} = E(1) \#_{F=S^1 \times m} S^1 \times$$



Want to calculate its SW-inv't

= Hoste

$$E(1)_L = E(1) \#_{F=S^1 \times m} S^1 \times s(\quad)$$

$s(L)$ = sewn-up link complement

$\ni \quad H_1(s(L)) = \mathbb{Z} \oplus \mathbb{Z}$

Proof of knot surgery thm $\Rightarrow$

Compute $SW_{E(1)_L}$ via macareña

Rule: $SW_{E(1)_{L_+}} = SW_{E(1)_{L_-}} + (t-t^{-1})^2 \, SW_{E(1)_{L_0}}$

links

knot

Use rule to calculate $SW_{E(1)_{\Lambda,0}} = t^{-1}-t$

$\Rightarrow$ for $\frac{1}{n}$ - surgery on Λ

$SW_{E(1)_{\Lambda,\frac{1}{n}}} = n\,(t^{-1}-t)$ $\Rightarrow$ ∞ family

(Can see all are homeo.)

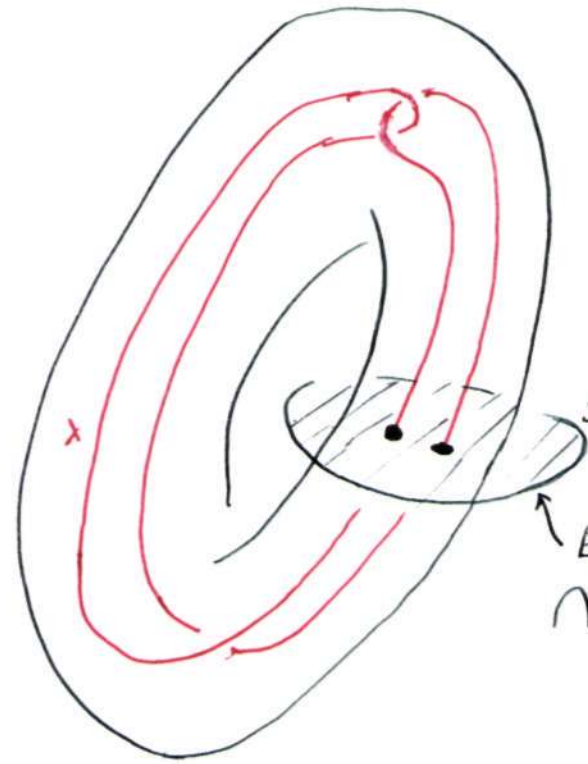
Λ is the dial

Dream - blow down $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$

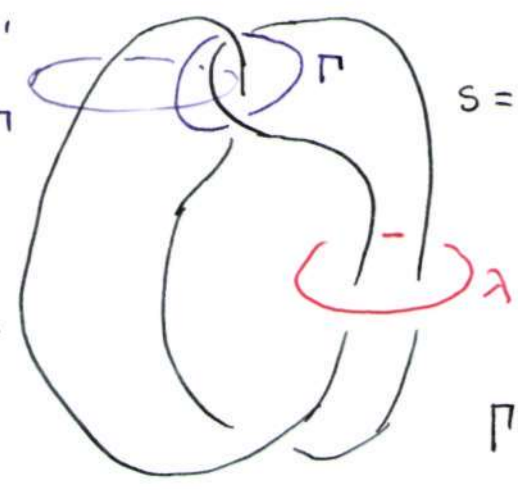
to find $\Lambda \subset \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$

then turn the dial

Dream can't come true



Alternative approach :



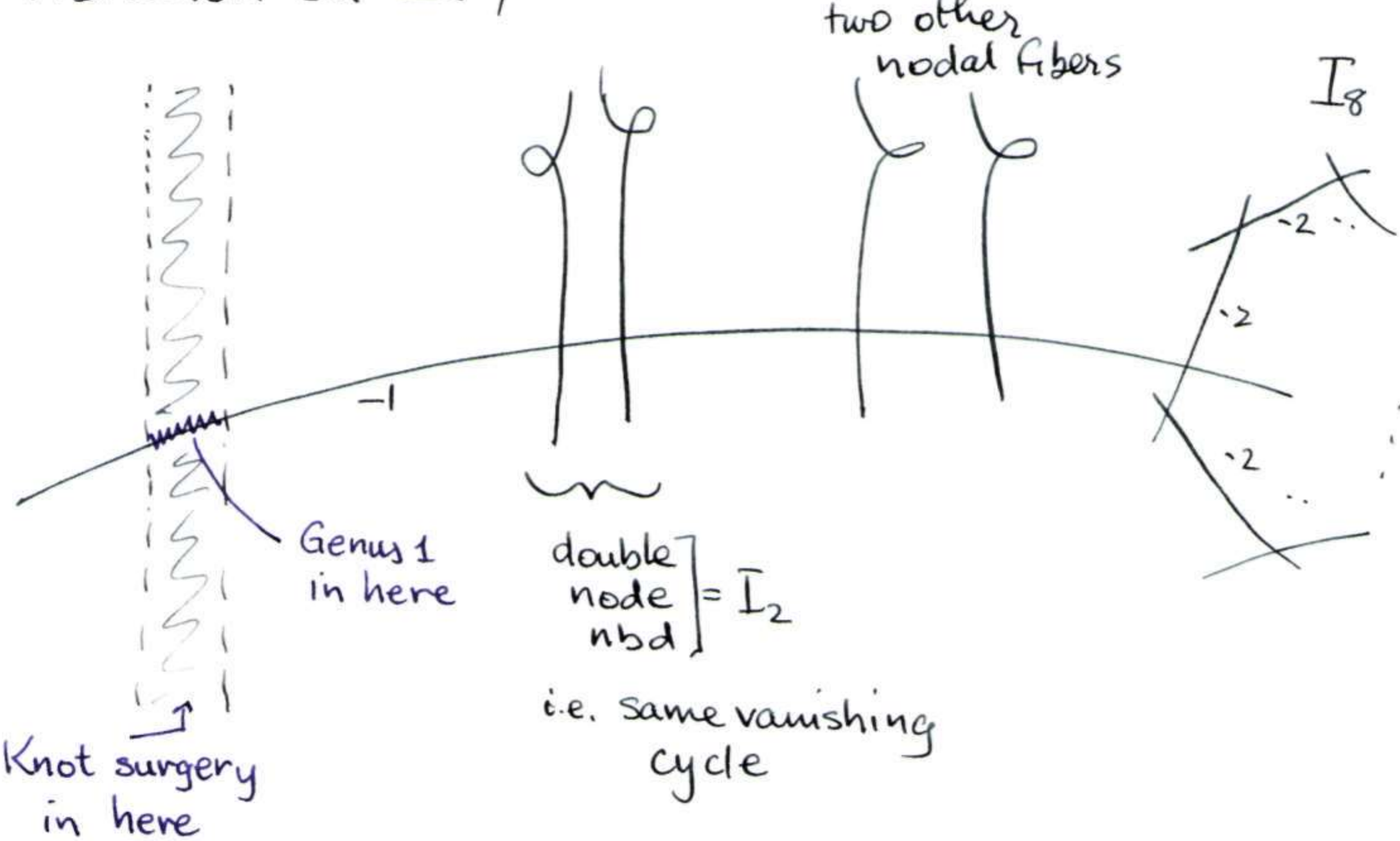
s = unknot (with visible genus 1 Seifert surface)

$\Gamma' + 2$ mends to s

bound punctured disk

$E(1) = E(1)_{K=S}$ - Genus one pseudosection

Fibration on $E(1)$

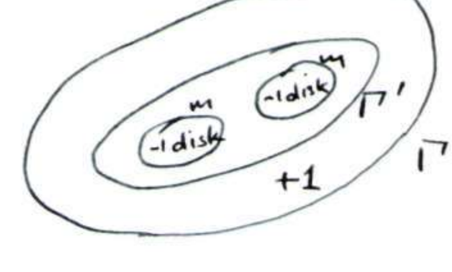


Freedom in knot surgery - gluing only restricted by longitude$(K) \longleftrightarrow \partial D^2$.

Can glue meridian m to van. cycle of double node nbd.

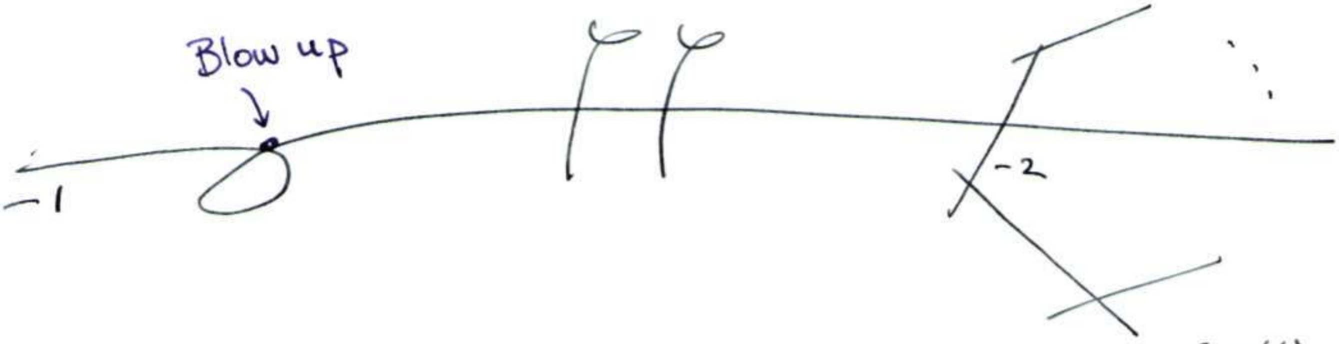
Then

Δ =



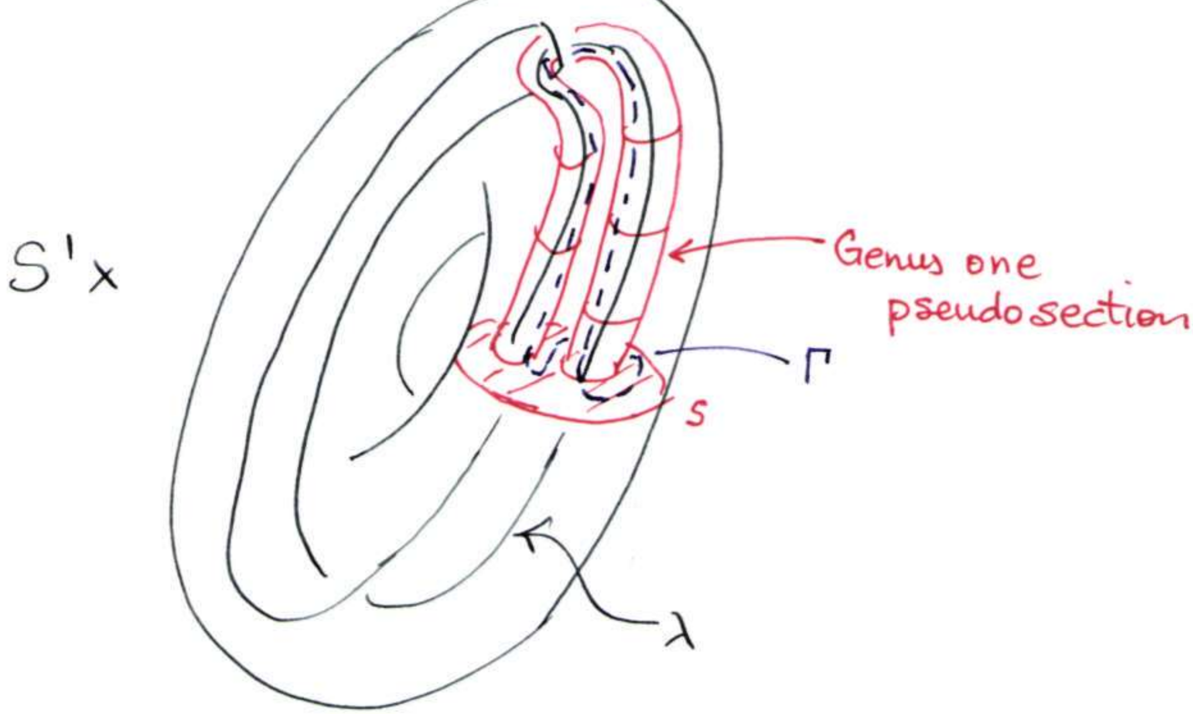
$\Rightarrow$ Γ bds disk with Δ rel framing $= -1$.

Ambient surgery wrt Δ turns pseudosection into immersed S^2



$\subset E(1)$
$= E(1)_S$

Λ is disjoint from all this :



Blow up double pt of immersed pseudosection

Get

$\subset \mathbb{CP}^2 \# 10\, \overline{\mathbb{CP}}^2$ with $\Lambda = S^1 \times \lambda$ in its complement

$\Big\downarrow$ rational blowdown

Get mfd R homeo to $\mathbb{CP}^2 \# 8\, \overline{\mathbb{CP}}^2$

$SW_R = 0$ (easy calculation)

[With work, can see $R \cong \mathbb{CP}^2 \# 8\, \overline{\mathbb{CP}}^2$]

$\Lambda \subset R$. Now spin the dial –

$\frac{1}{n}$ - surgeries on Λ give ∞ family as before.

Same techniques work for $k = 5, 6, 7$

How to find other useful nullhomologous tori

One way – Reverse engineering

Start with 4-mfd Y with:

- $SW_Y \neq 0$
- $b_1(Y) = 1$
- $\exists$ "H_1-essential torus" i.e. T of self-$\cap = 0$ with loop γ on it generating $H_1(Y)$

Surger T, killing γ. Also kills a hyp. pair in H_2.

Get X, $H_1(X) = 0$ & $H_2(Y) = H_2(X) \oplus (\text{hyp pair})$

Λ = Core torus of surgery in X

Λ nullhomologous "0"-surgery on it $\rightsquigarrow Y$

$\Rightarrow$ ∞'ly many distinct smooth mfds with same homology as X.

<u>Ex.</u> $Y = \mathrm{Sym}^2(\Sigma_3)$ perform process iteratively

Get ∞'ly many mfds with $H_* = H_*(\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2)$

$Y = \Sigma_2 \times \Sigma_2$ . . .

Get ∞'ly many mfds with $H_* = H_*(S^2 \times S^2)$.