# Reverse Engineering

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Four Dimensional Topology
Hiroshima University

Joint work with Ron Stern

*Things which are seen are temporal,*
*but the things which are not seen are eternal.*
B. Stewart and P.G. Tait

### Wild Conjecture

Every smooth simply connected 4-manifold has infinitely many distinct 4-manifolds which are homeomorphic to it.

The goal of this lecture —
Discuss a technique which can be used to study this conjecture

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Find a “dial” (figuratively) to turn
to change the smooth structure at will.

This “dial”: Surgery on nullhomologous tori

# Nullhomologous Tori

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This "dial": Surgery on nullhomologous tori

# Knot Surgery

$K$: Knot in $S^3$, $T$: square 0 essential torus in $X$

**Definition**

$X_K = (X \setminus N_T) \cup (S^1 \times (S^3 \setminus N_K))$

**Facts about knot surgery**

- If $X$ and $X \setminus T$ both simply connected; so is $X_K$.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

**Conclusion**

- If $X$, $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to $X$.

*e.g.* $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

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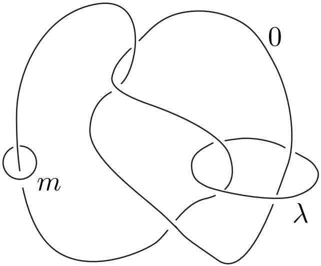
## Relation of knot surgery to nullhomologous tori — proof of Knot Surgery Theorem

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$X_K = X \#_{T = S^1 \times m} S^1 \times$ 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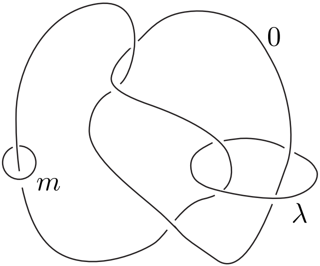


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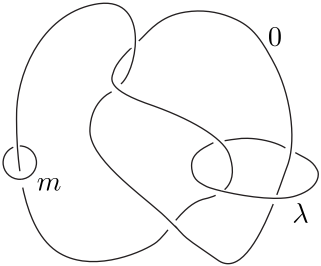


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# The Morgan, Mrowka, Szabo Formula

Describes how surgery on a torus changes the Seiberg-Witten invariant

$T$: torus in $X$ with self-intersection $= 0$ Nbd $= S^1 \times S^1 \times D^2$
Do $S^1 \times (p/q)$ - surgery (precise description below) to get $X'$

Roughly

$$\mathcal{SW}_{X'} = p\,\mathcal{SW}_X + q\,\mathcal{SW}_{X_0}$$

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# An Example: Some Smooth Structures on $E(1)$

$$E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$$

Elliptic surface $F$: fiber (torus of square 0) $N_F = S^1 \times S^1 \times D^2$

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Nullhomologous torus in $E(1)$
$=$ Whitehead double of fiber

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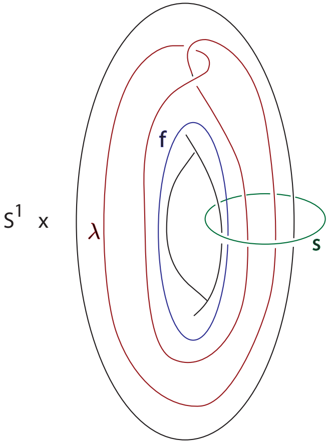


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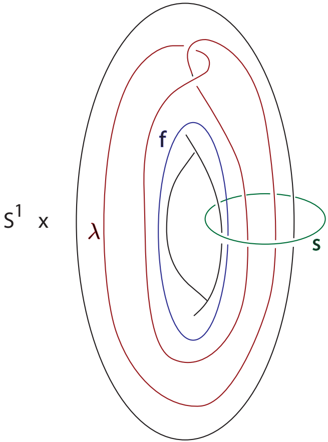


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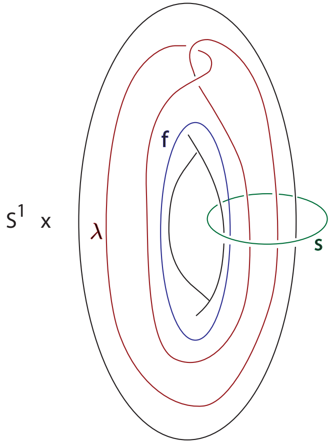


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# Smooth Structures on $E(1)$, cont.

$$\mathcal{SW}_{E(1)} = 0 \implies \mathcal{SW}_{E(1)_{\Lambda,1/n}} = n\,\mathcal{SW}_{E(1)_{\Lambda,0}}$$

(by Morgan, Mrowka, Szabo)

$E(1)_{\Lambda,0}$ obtained by killing longitude of $\lambda$ by surgery
Has $b_1 = 1$ and $b^+ = 2$

Whitehead link symmetry $\implies$

Achieve this in $E(1)$ directly by knot surgery on $s =$ unknot.

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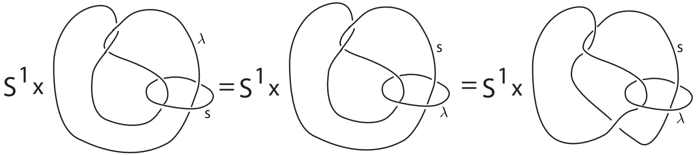


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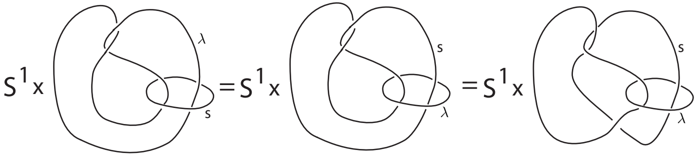


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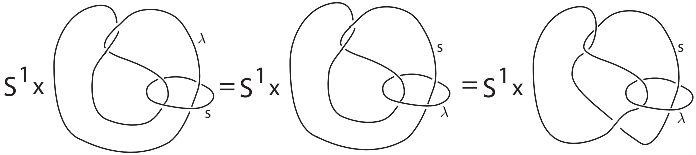


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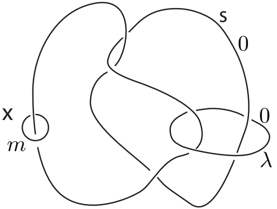
# 0-Surgery on $\Lambda$

Accomplished by gluing $S^1$ x 

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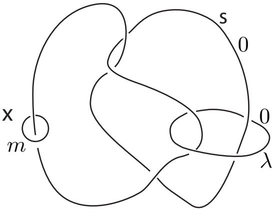
Accomplished by gluing $S^1 \times$ 

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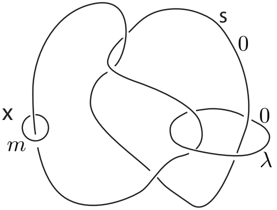
Accomplished by gluing $S^1 \times$  into E(1)



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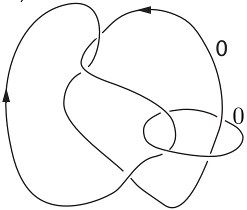
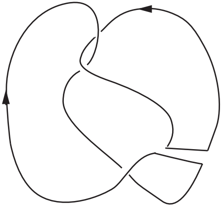




= s (  )

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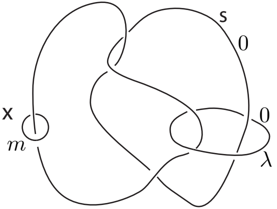


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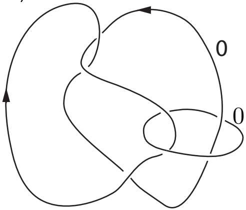


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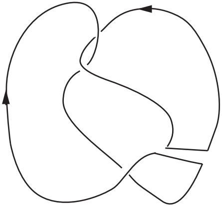

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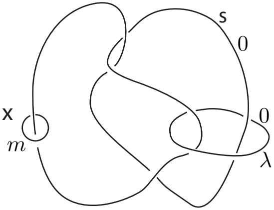


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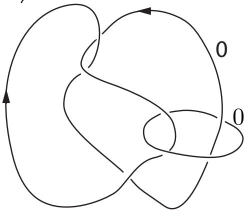
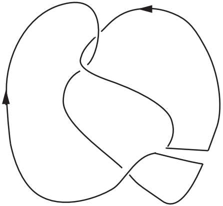



= s (  )

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# An Infinite Family of Smooth Structures on $E(1)$

$\mathcal{SW}_{E(1)_{\Lambda,0}}$ calculated by macarena moves on $L$

Can use this to calculate $\mathcal{SW}_{E(1)_{\Lambda,0}} = t^{-1} - t$

$\implies 1/n$ - surgeries on $\Lambda$ give manifolds homeo to $E(1)$ and

$$\mathcal{SW}_{E(1)_{\Lambda,1/n}} = 1 \cdot \mathcal{SW}_{E(1)} + n\,\mathcal{SW}_{E(1)_{\Lambda,0}} = n(t^{-1} - t)$$

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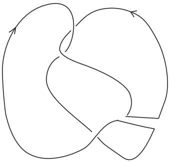
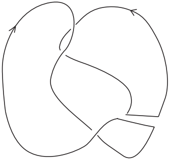
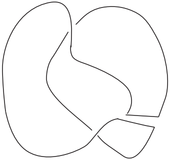
$\mathcal{SW}($  $) = \mathcal{SW}($  $) - (t - t^{-1})^2\ \mathcal{SW}^{-}($  $)$

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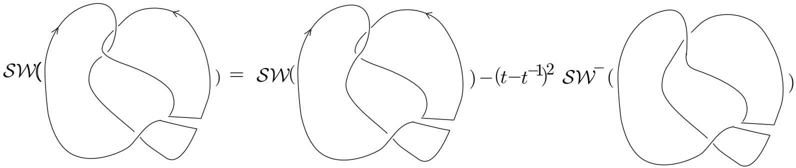


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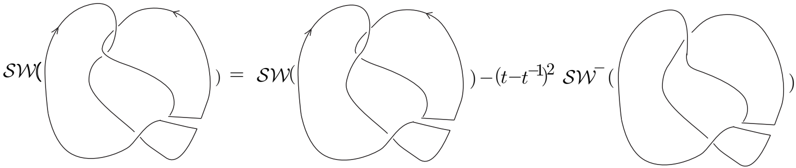


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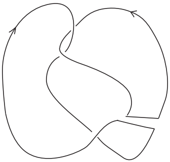
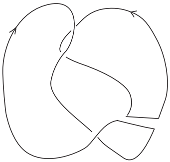
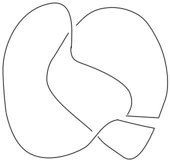
$\mathcal{SW}($  $)$ $=$ $\mathcal{SW}($  $) - (t - t^{-1})^2\ \mathcal{SW}^{-}($  $)$

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$$\mathcal{SW}_{E(1)_{\Lambda,1/n}} = 1 \cdot \mathcal{SW}_{E(1)} + n\,\mathcal{SW}_{E(1)_{\Lambda,0}} = n(t^{-1} - t)$$

$\Longrightarrow$ infinite family

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Procedure to insure their existence:

1. Find model manifold $M$ with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $\mathcal{SW} \neq 0$.
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# Surgery on tori

$T = \alpha \times \beta$: square 0 torus in $M$. $T^3 = \partial N_T$.

$S^1_\alpha$, $S^1_\beta$ loops in $T^3$ such that $S^1_\alpha \sim \alpha$ and $S^1_\beta \sim \beta$ in $N_T$

$$\partial N_T = S^1_\alpha \times S^1_\beta \times \partial D^2 \qquad \mu = \partial D^2$$

$p/q$-surgery on $T$ w.r.t. $\beta$ means:

$$M_{T,\beta}(p/q) = (M \smallsetminus N_T) \cup_\varphi (S^1 \times S^1 \times D^2)$$
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such that $\varphi_*[\partial D^2] = q[S^1_\beta] + p\mu$ in $H_1(\partial(M \smallsetminus N_T))$

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This operation does not change $e(M)$ or $\mathrm{sign}(M)$.

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(a) $\longrightarrow$ (b) reduces $b_1$ by 1 and decreases $H_2$ by a hyperbolic pair.

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Iterate this construction to kill $H_1(M)$.

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Iterate this construction to kill $H_1(M)$.

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$X$: symplectic manifold   $T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Auroux, Donaldson, Katzarkov)

If $S^1_\beta$ = Lagrangian pushoff, $X_{T,\beta}(\pm 1)$: symplectic mfd

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# Families

- The $\mathcal{SW}$ condition
  If $M$ is symplectic and surgery tori are Lagrangian and we do $(\pm 1)$-surgeries with respect to the Lagrangian framings, each resultant manifold will be symplectic and have $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold $X$
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Model Manifold $= Sym^2(\Sigma_3)$

Has the same $e$ and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has disjoint Lagrangian tori carrying basis for $H_1$.

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# References for Constructions


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# Constructing Model Manifolds

## Chern number and Holomorphic Euler number

For a symplectic 4-manifold, $X$,

$$c_1^2(X) = \frac{1}{4}(e(X) + \mathrm{sign}(X)) \text{ and } \chi(X) = 3\,\mathrm{sign} + 2\,e(X)$$

## Fiber Sums

If $X'$, $X''$ are symplectic with symplectic submanifolds $\Sigma'$, $\Sigma''$ of square 0 and same genus $g$, the fiber sum $X = X' \#_{\Sigma' = \Sigma''} X''$ is again symplectic, and

- $c_1^2(X) = c_1^2(X') + c_1^2(X'') + 8(g-1)$
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## Model Manifolds

Constructed from fiber sums where $g = 2$.

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# Many Model Manifolds

## Basic Pieces: $X_0, X_1, X_2$

$X_0 = T^2 \times \Sigma_2, \quad c_1^2(X_0) = 0,\ \chi(X_0) = 0$

$\Sigma = \mathrm{pt} \times \Sigma_2.$

$X_1 = T^2 \times T^2 \# \overline{\mathbb{CP}}^2, \quad c_1^2(X_1) = -1,\ \chi(X_1) = 0$

In $T^2 \times T^2$, call first torus $T_1$ and second $T_2$.
$2T_1$ also represented by a torus. $2T_1$ intersects $T_2$ in two points.
Blow up one and smooth the other. Get $\Sigma$: genus 2, square 0.
$\Sigma$ homologous to $2T_1 + T_2 - 2E$.

$X_2 = T^2 \times T^2 \# 2\,\overline{\mathbb{CP}}^2, \quad c_1^2(X_2) = -2,\ \chi(X_1) = 0$

In $T^2 \times T^2$, blow up $T_1 + T_2$ twice. Get $\Sigma$: genus 2, square 0
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In $S^2 \times T^2$ there is an embedded torus $T'$ representing $2T^2$.
Consider configuration $T' + T^2 + S^2$ which has 3 double points.
Blowup one double point on $T'$ and smooth the other two double points. Then blow up at two more points on the result.
Get $\Sigma$: genus 2, square 0 homologous to $3T^2 + S^2 - 2E_1 - E_2 - E_3$.

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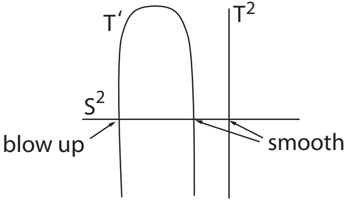
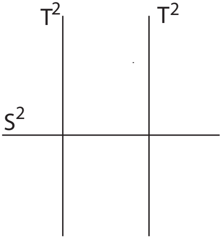


# Many Model Manifolds
## Basic Pieces: $X_4$

$X_4 = S^2 \times T^2 \# 4\,\overline{\mathbb{CP}}^2, \quad c_1^2(X_0) = -4,\ \chi(X_0) = 0$

In $S^2 \times T^2$ consider configuration with 2 disjoint copies of $T^2$ and one $S^2$. Smooth the double points and then blow up at 4 points to get $\Sigma$ homologous to $2T^2 + S^2 - E_1 - E_2 - E_3 - E_4$.
$\Sigma$ has genus 2 and square 0.



# Many Model Manifolds

Model for $b^+ = 1$, $b^- = k, k = 1, \ldots, 8$
$(c_1^2 = 9 - k,\ \chi = 1)$

$M_k = X_i \#_\Sigma X_j$, where $i + j = k - 1$

$$c_1^2(M_k) = c_1^2(X_i) + c_1^2(X_j) + 8 = 9 - k$$

$$\chi(M_k) = \chi(X_i) + \chi(X_j) + 1 = 1$$

Enough Lagrangian tori to surger to kill $H_1 \Longrightarrow$ infinte family

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## A particular example: $b^- = 1$

$$M_1 = X_0 \#_\Sigma X_0 = (T^2 \times \Sigma_2) \#_{\Sigma_2} (T^2 \times \Sigma_2) \cong \Sigma_2 \times \Sigma_2$$

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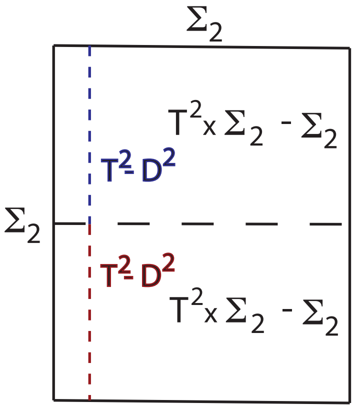


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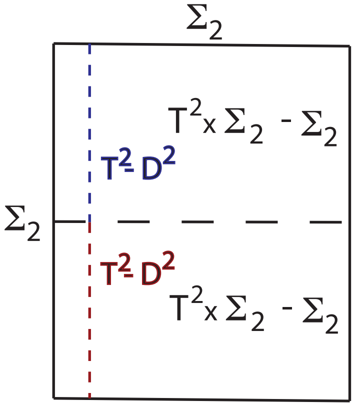


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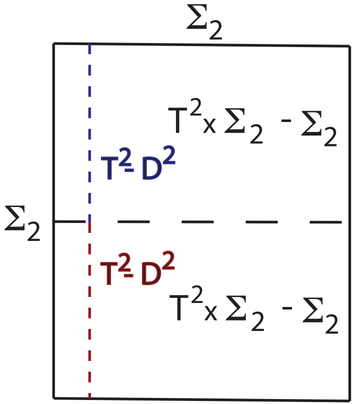


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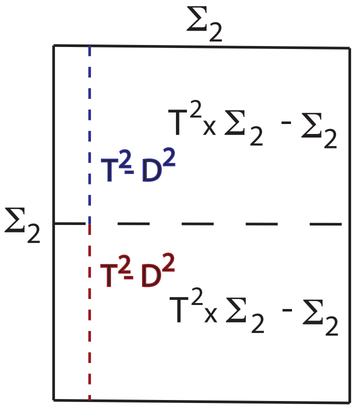


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Question: What about $M_3' = X_1 \#_\Sigma X_1$?

A Challenge

In $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ find a nullhomologous torus so that surgeries on it give the known fake examples.

Santeria Surgery

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