$b^+ = 1$ simply connected 4-mfds

(Joint work w. Ron Stern)

Std exs: rational and rational ruled surfaces

$\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, $k \geq 0$, and $S^2 \times S^2$

Goal Construct mfds homeo not diffeo to these

History: Donaldson (~1985), Dolgachev surfaces ($k=9$)

Kotschick (~1988) Barlow surface ($k=8$)

Jongil Park (~2004) ($k=7$)

& with Yongnam Lee - as cx surface (2007)

Combined work of Park, Stipsicz-Szabo, F-Stern

$k=5,6$ + ∞'ly many ex's.

J. Park & students - cx surfaces for $k=5,6$

- Very small 4-mfds -

Akhmedov-D.Park / Baldridge-Kirk $k=3$ (2007)

| ∞'ly many F-Stern

($k=4$)

$k=2$ Akhmedov-Park - work in progress

Where we stand - $k=0,1$?

$k \geq 10$ & minimal ?

Knot Surgery

Remove torus of self-int $=0$ / Replace with $S^1 \times (S^3 \setminus \text{nbd(knot)})$

- If X & $X \setminus$ torus s.c. then get X_K homeo X

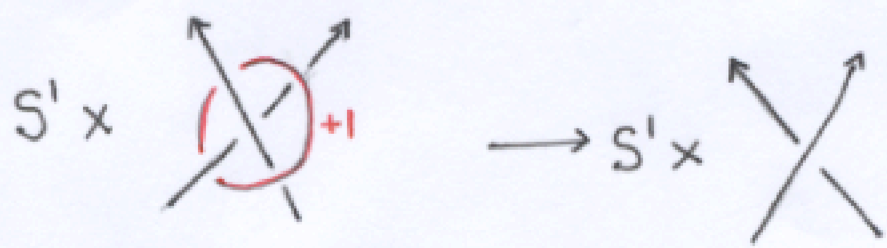
& $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$ (F-Stern)

- So K nontriv & torus essential $\Rightarrow$ ∞'ly many mfds homeo not diffeo X.

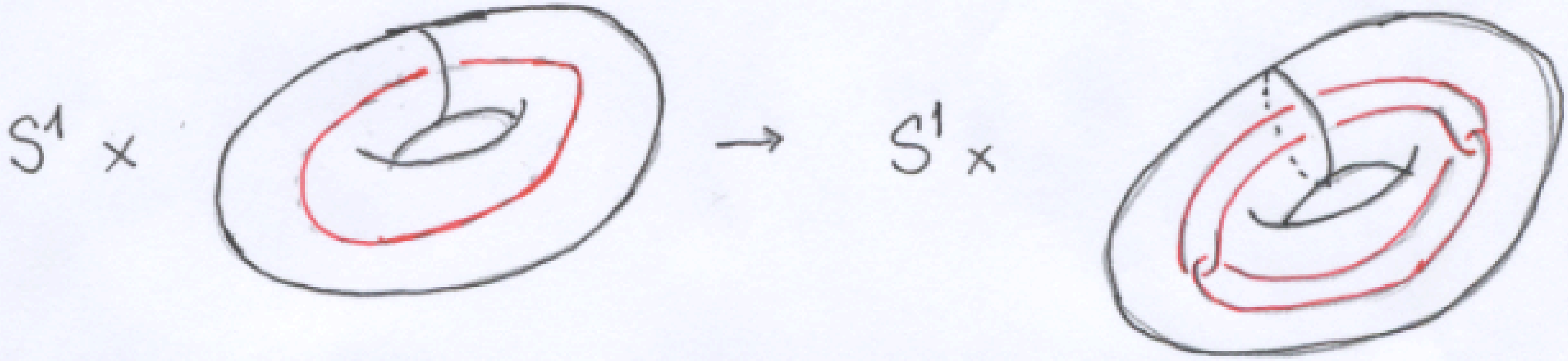
What about mfds with no such tori?

Nullhomologous tori

- Used to prove knot surgery theorem:



Bing doubling produces pairs of nullhomologous tori:

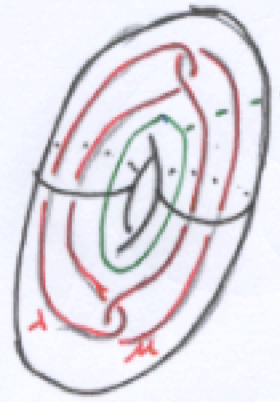


- Important theorem of Morgan-Mrowka-Szabo
"$SW_{X_{p/q}} = p\, SW_X + q\, SW_{X_0}$"

for $S^1 \times (p/q)$ - Dehn surgery on torus of square 0 in X.

Simple Example

$E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2$ elliptic surface, $F \cong T^2$: fiber

Nbd(F): $S^1 \times$ 

$F = S^1 \times f$

Bing double
$S^1 \times \lambda$, $S^1 \times \mu$

What is result of -1 surgery on $S^1 \times \lambda$ and $\frac{1}{n}$-surgery on $S^1 \times \mu$?

-1 on $S^1 \times \lambda \rightsquigarrow$ $S^1 \times$ 

$S^1 \times \nu$ = Whitehead double of F

$SW_{E(1)} = 0$ so M-M-Sz-formula $\Rightarrow SW_{\text{Result}} = n \underbrace{SW_{E(1)_{\text{0-surgery on } S^1\times\nu}}}_{\neq 0}$

This nullhomologous torus "useful" provided

Can calculate $SW_{E(1)_0} = t^{-1} - t$

$\Rightarrow$ get ∞'ly many mfds homeo $E(1)$.

- Can't blow down to get smaller mfds since section = exc. curve $\cap$'s $S^1 \times \nu$ in 2 pts.

Double node surgery (F-Stern) gets around this for $k = 5, 6, 7, 8$.

Finding useful nullhomologous tori by Reverse Engineering

X_1: sympl. 4-mfd w/ $b_1 = 1$

T_1: primitive Lagr. torus "carrying b_1"

decr. H_2 by hyp. pair $\Big\downarrow$ ± 1 surgery (wrt Lagrangian framing)

X: sympl 4-mfd with $b_1 = 0$

$\cup$

T: nullhomologous torus

0-surgery on $T \rightsquigarrow X_1$ & $SW_{X_1} \neq 0$.

- Can iterate this construction:

  Start with X_n with $b_1 = n$ carried by n disjoint prim. Lagr. tori ...

Note $e(X_n) = e(X)$, $\operatorname{sign}(X_n) = \operatorname{sign}(X)$

Thm (F-Stern) Surgeries on $T \subset X$ lead to ∞'ly many distinct mfds w/ same H_* as X.
(If all π_1's $= 0$ then homeo to X.)

X_n: model for X

Ex $\quad \operatorname{Sym}^2 \Sigma_3 = $ Model for $\mathbb{P}^2 \# 3\overline{\mathbb{P}}^2$

$\quad \Sigma_2 \times \Sigma_2 = \quad \cdots \quad S^2 \times S^2$

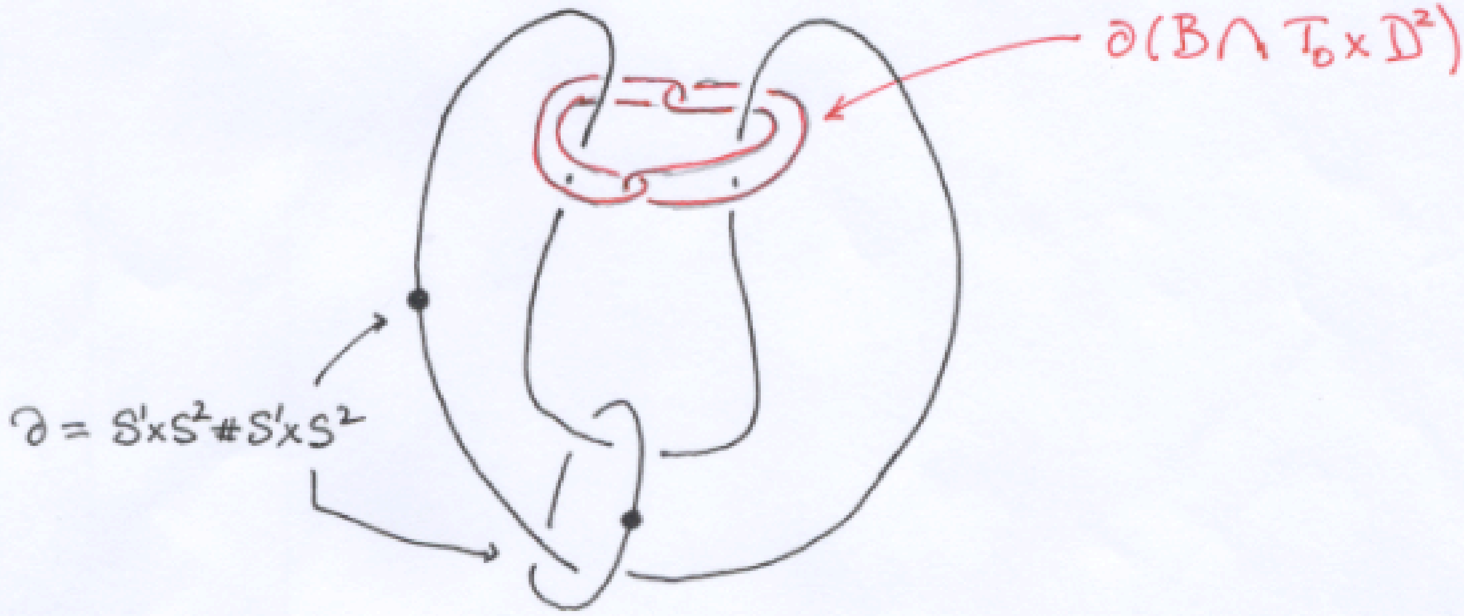
Santeria Surgery

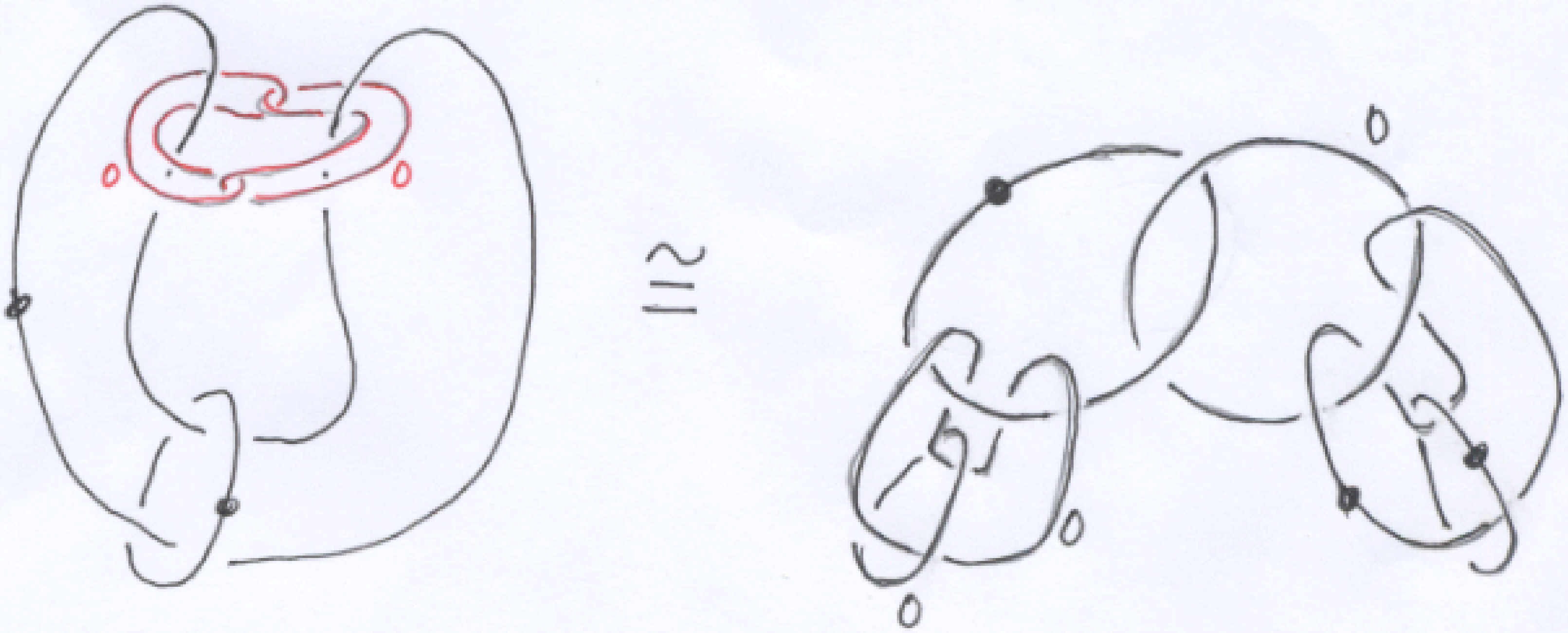
How to find useful nullhomologous tori in given X.

B = Bing double of core torus in $T^2 \times D^2$.

$T_0 = T^2 \setminus D^2$ $\qquad$ $B \cap (T_0 \times D^2)$ = pair of punctured tori

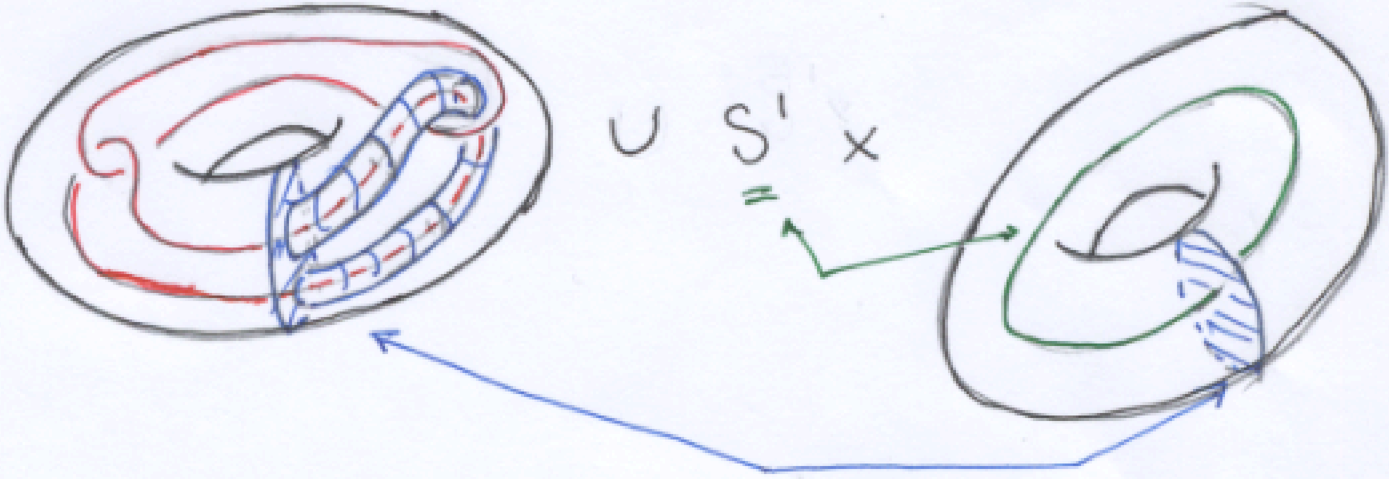
$$\partial(T_0 \times D^2) = S^1 \times S^2 \# S^1 \times S^2$$

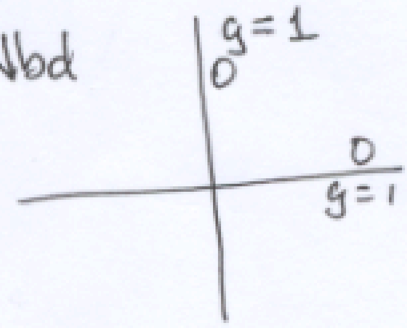


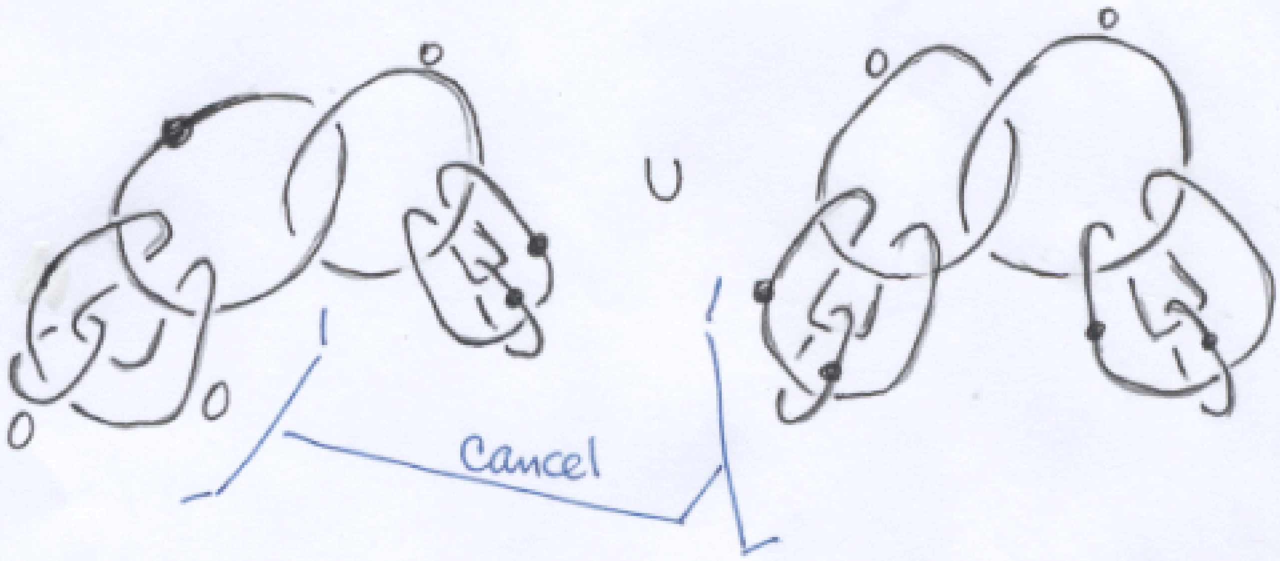
Set A = 

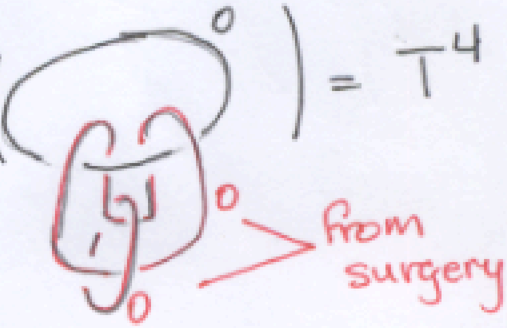
B C A

$A =$ complement of pair of transverse tori in $T^2 \times S^2$

$S^1 \times$ 

Idea: $A \cup$ Nbd 

$=$ 

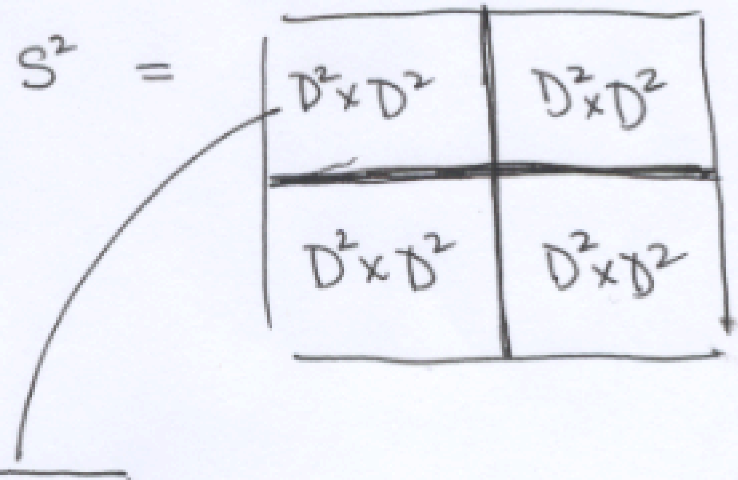
Surgery on $B \subset A \subset T^2 \times S^2 = S^1 \times \partial($  $) = T^4$

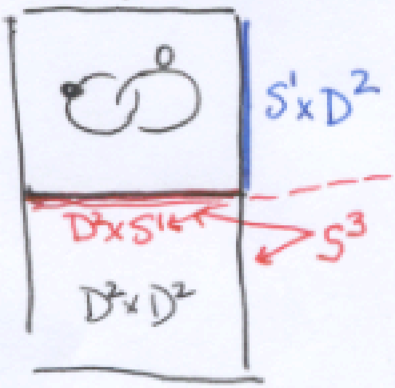
Removing "axes"

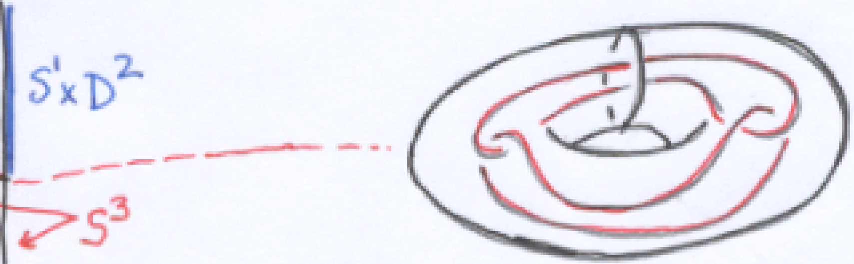
$$A \xleftrightarrow{\text{2 surgeries}} T_0 \times T_0$$

Application to Santeria Surgery

Ex. $S^2 \times S^2 =$

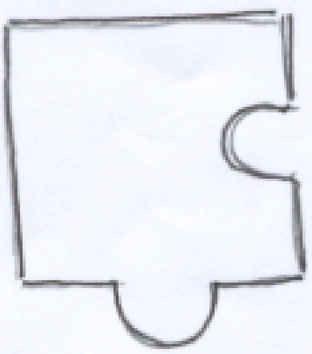




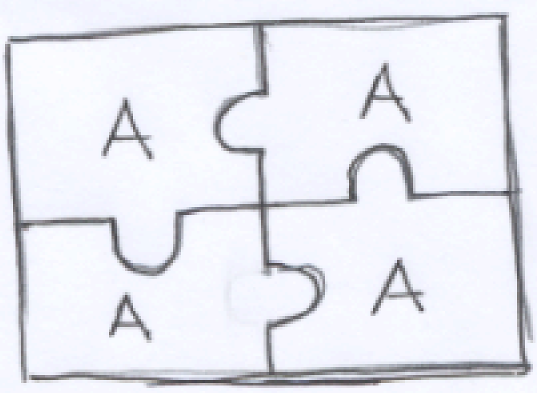


Link bds disjoint D^2's pushed into lower 4-ball.

$\Rightarrow$ Can attach 2-hdles scooped from bottom.
($\Rightarrow$ attaches 1-hdles to bottom 4-ball.)

 $=$ 

$= A$

$\Rightarrow S^2 \times S^2 =$ 

$\Downarrow \xrightarrow[\text{surgeries}]{\text{8 Santeria}}$ Homology $S^2 \times S^2$'s

$\pi_1 = ?$

8 surgeries

Rev Eng (8 times)

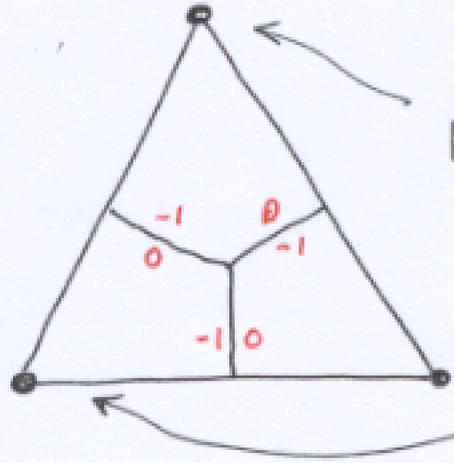
$\Rightarrow$

$\Sigma_2 \times \Sigma_2 =$

$T_0 \times T_0$	$T_0 \times T_0$
$T_0 \times T_0$	$T_0 \times T_0$

$\mathbb{P}^2 \# 3\overline{\mathbb{P}}^2$

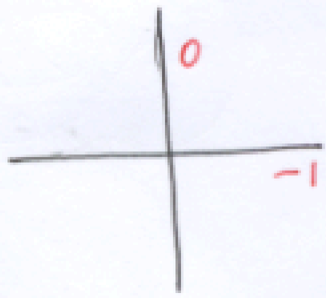
$\mathbb{P}^2 =$



Nbds of 3 std $\mathbb{CP}^0$'s in homog coord's each $\cong B^4$

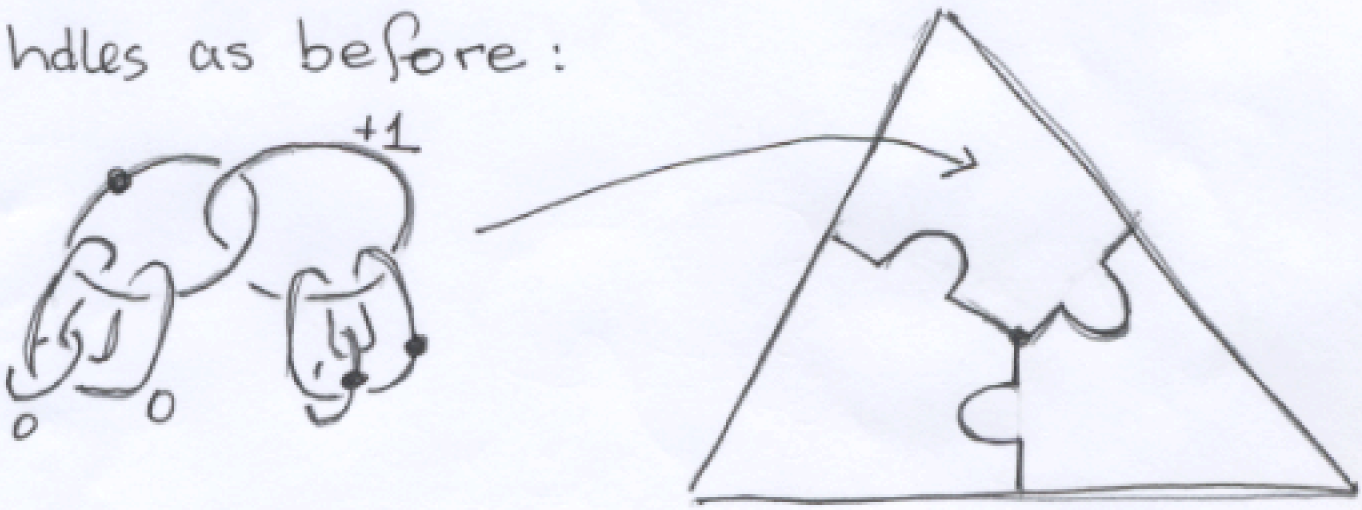
View as Symington 3-fold sum: Each 4-ball is

$\mathbb{F}_1 \setminus \{\text{fiber} \cup \text{-'ve section}\}$

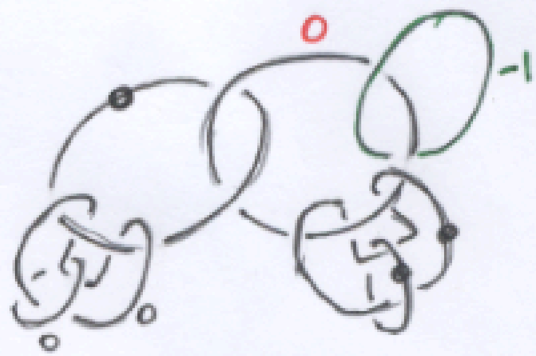


Complement =

Trade hdles as before:



Blow up each piece to find A



Pair of surgeries

Compl. of T_1 & $T_2 - E$ in $T^2 \times T^2 \# \overline{\mathbb{P}}^2$

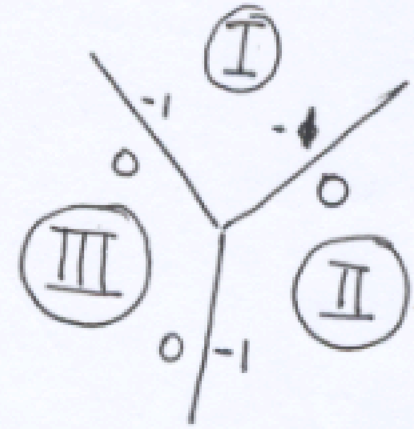
Also get 3 fold sympl sum

$\Rightarrow$ sympl. (Presumably $Sym^2(\Sigma_3)$)

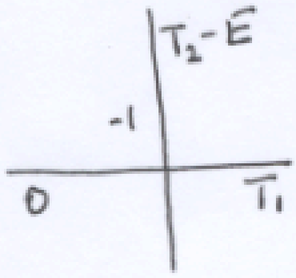
Get exotic $\mathbb{P}^2 \# 3\overline{\mathbb{P}}^2$'s via Santeria Surgery

$\underline{\mathbb{P}^2 \# 2\overline{\mathbb{P}}^2}$ via 3-fold sum

Sympl. model mfd $\underline{M}$ =



II:



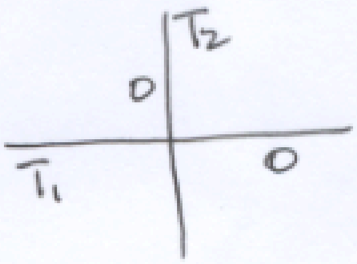
$\subset T^4 \# \overline{\mathbb{P}}^2$

as above

$\xleftrightarrow{\text{2 surgeries}}$

$e=2$

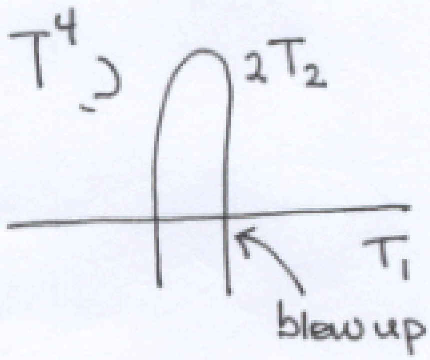
III:



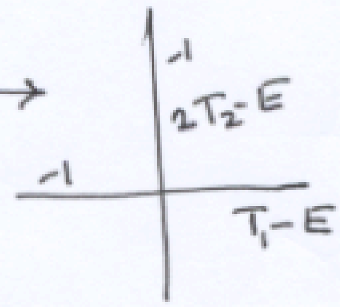
$\subset T^4$ $\xleftrightarrow{\text{2 surgeries}}$ A

$e=1$

I:



$\rightarrow$

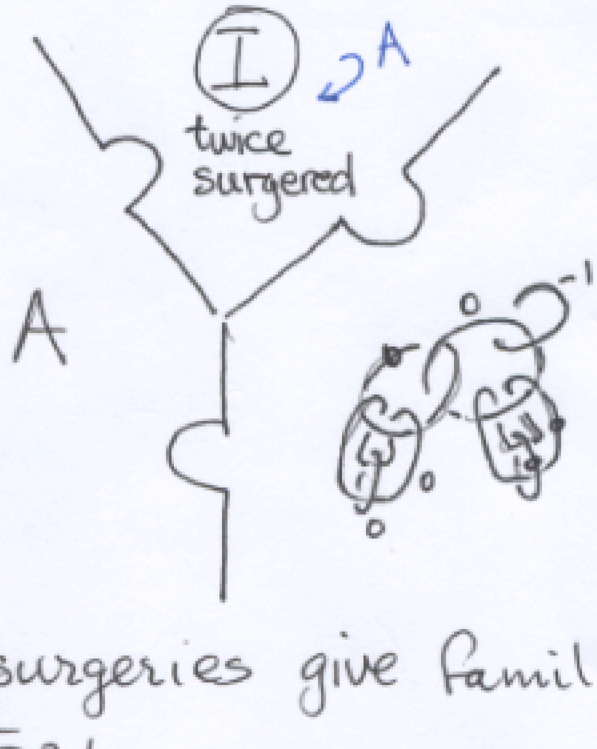


$\subset T^4 \# \overline{\mathbb{P}}^2$

Note $T_0 \times T_0$

$e=2$

$\xrightarrow{\text{6 surgeries}}$



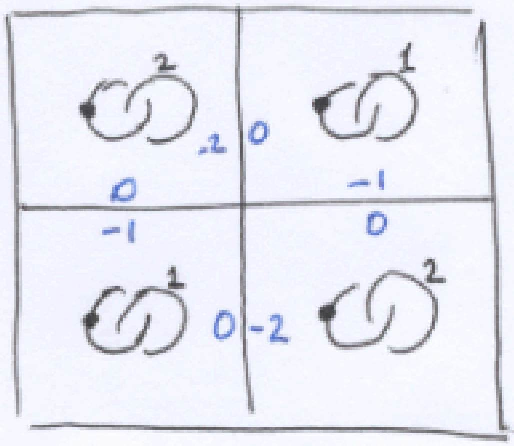
Note $e=5$, $b_1=0$

$\cong \mathbb{P}^2 \# 2\overline{\mathbb{P}}^2$?

6 nullhomologous surgeries give family of homology $\mathbb{P}^2 \# 2\overline{\mathbb{P}}^2$'s.

$\pi_1 = ?$

Teaser - b^+ even:



$\cong 2\mathbb{P}^2$

To get copies of A after trading hdles need to blow up 6 times.

Get 8 nullhomologous tori in $2\mathbb{P}^2 \# 6\overline{\mathbb{P}}^2$

What does "useful" mean? — No inv'ts.