Reverse-engineering families of 4-manifolds

Ron Fintushel
Michigan State University
June 18, 2007

Joint work with Ron Stern

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^i rk(H^i(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = $\text{sign}(X)$ = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^i rk(H^i(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = $\mathrm{sign}(X)$ = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^i rk(H^i(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;
  $\alpha \cdot \beta = (\alpha \cup \beta)[X]$
  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of

the canonical class of a complex surface.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of

the canonical class of a complex surface.

Geometric Topology Conference – Beijing 2007

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of

the canonical class of a complex surface.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
$$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of

the canonical class of a complex surface.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
$$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$
$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$
for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of

the canonical class of a complex surface.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of the canonical class of a complex surface.

Geometric Topology Conference – Beijing 2007

What do we know about smooth 4-manifolds?

Wild Conjecture

Every (simply connected) 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
- $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of the canonical class of a complex surface.

Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal
$c = 3\text{sign} + 2e$
$\chi_h = \frac{\text{sign}+e}{4}$

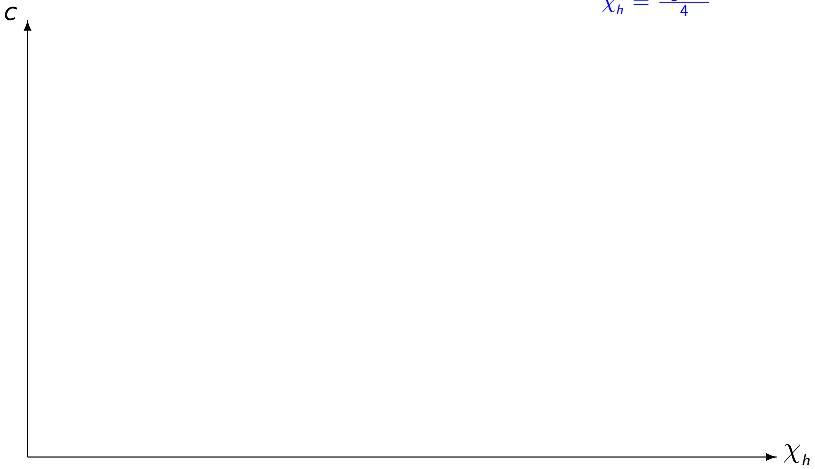
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$



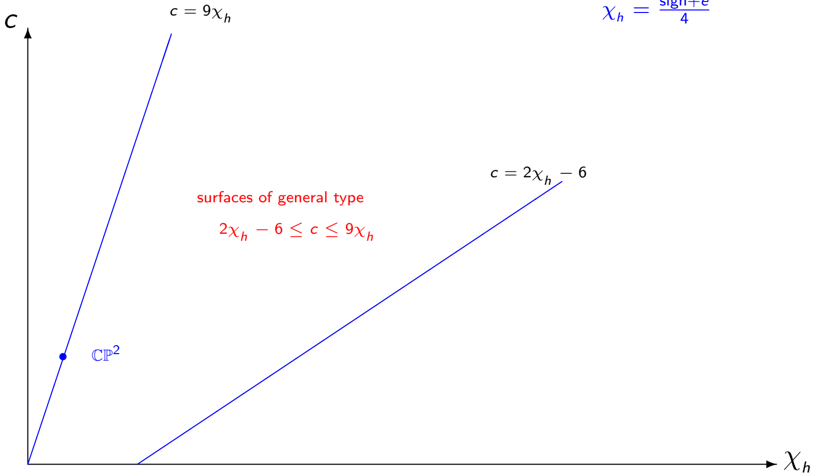
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal

$c = 3\mathrm{sign} + 2e$

$\chi_h = \frac{\mathrm{sign}+e}{4}$



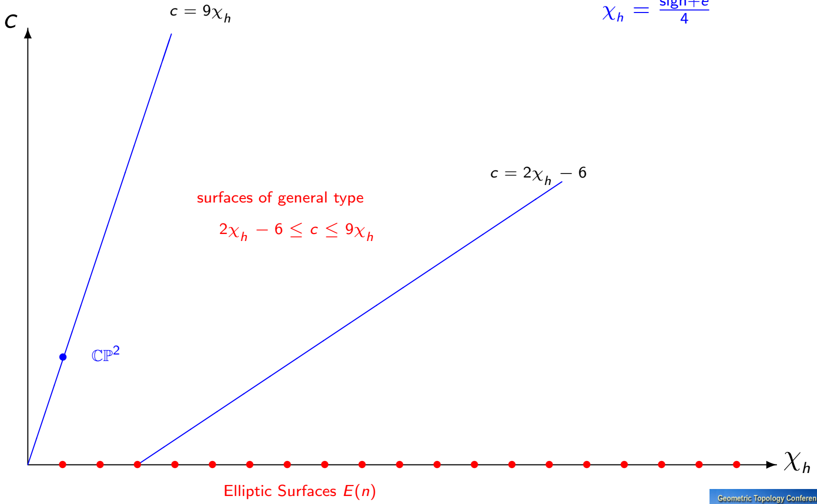
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal

$c = 3\text{sign} + 2e$

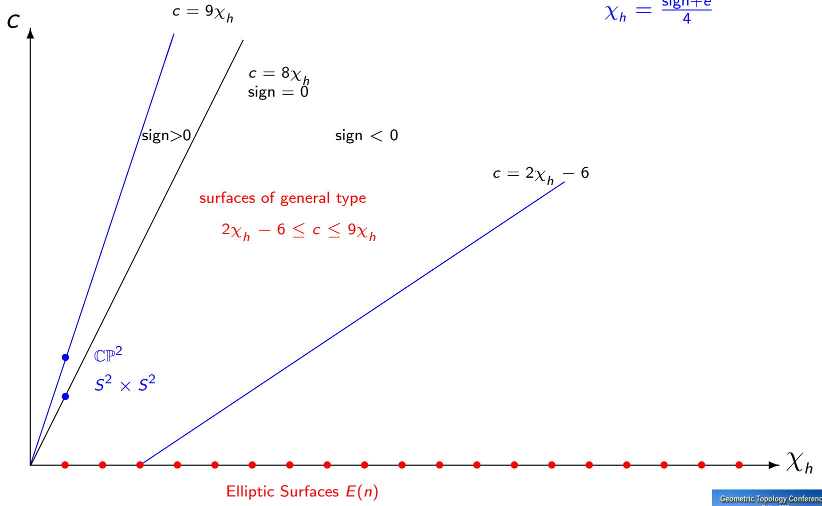
$\chi_h = \frac{\text{sign}+e}{4}$



Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal
$c = 3\text{sign} + 2e$
$\chi_h = \frac{\text{sign}+e}{4}$



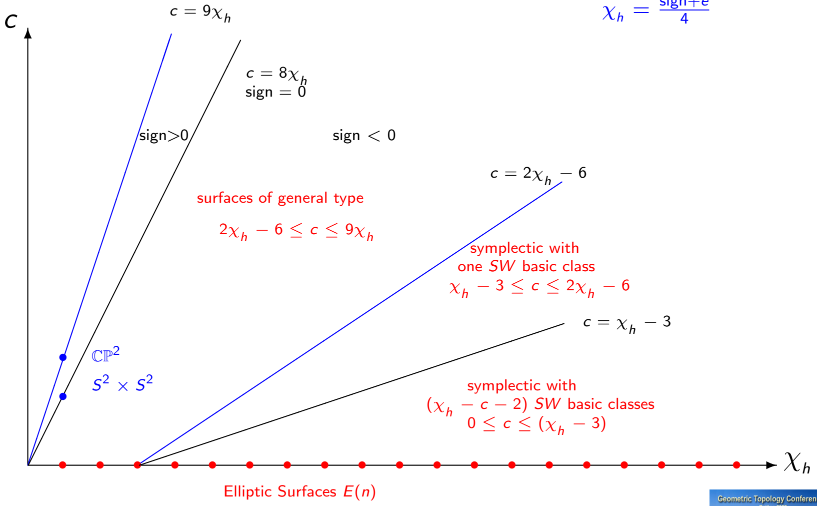
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$



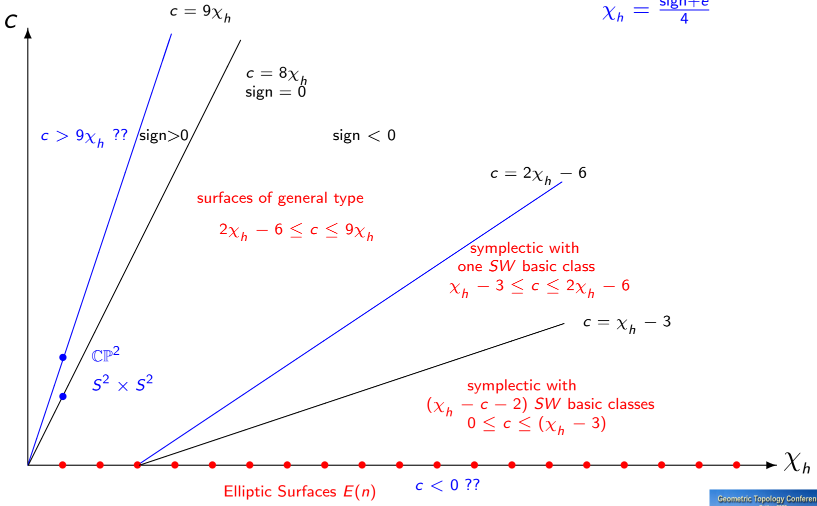
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$



Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

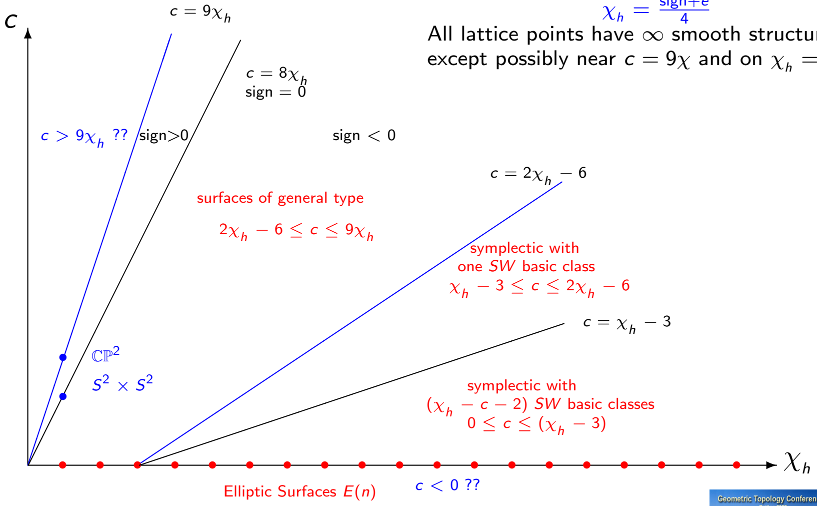
Geography

All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$



Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

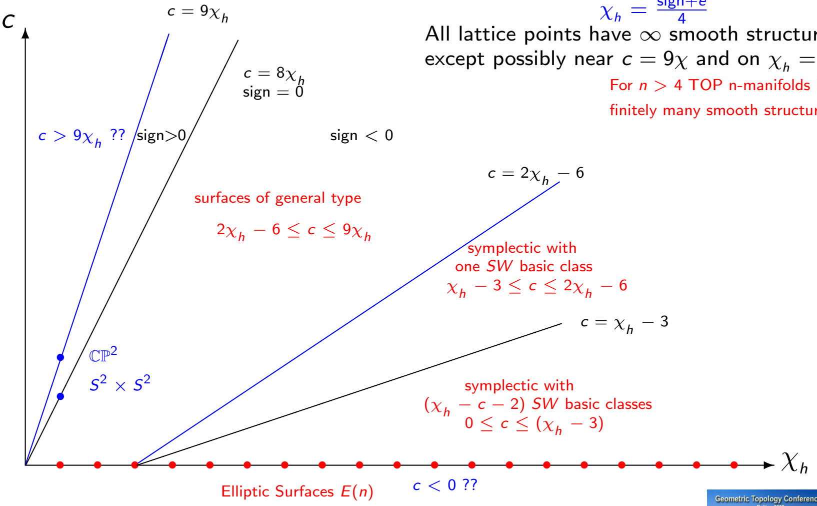
All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Geometric Topology Conference – Beijing 2007

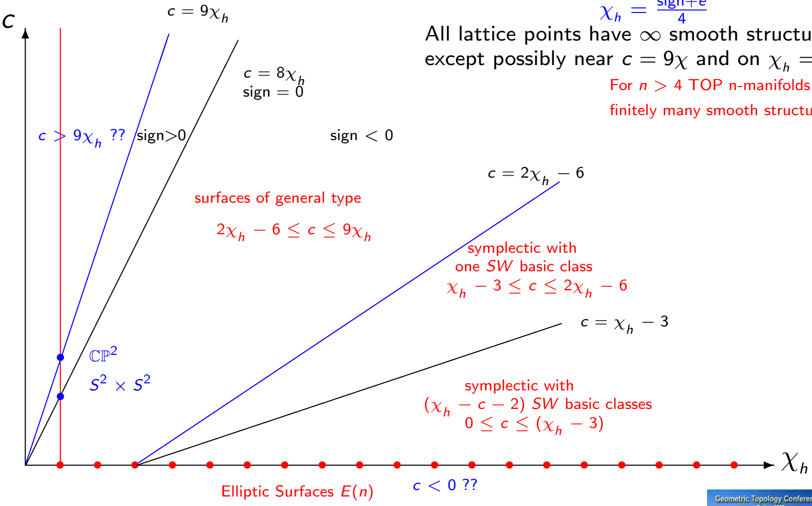
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal
$c = 3\text{sign} + 2e$
$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

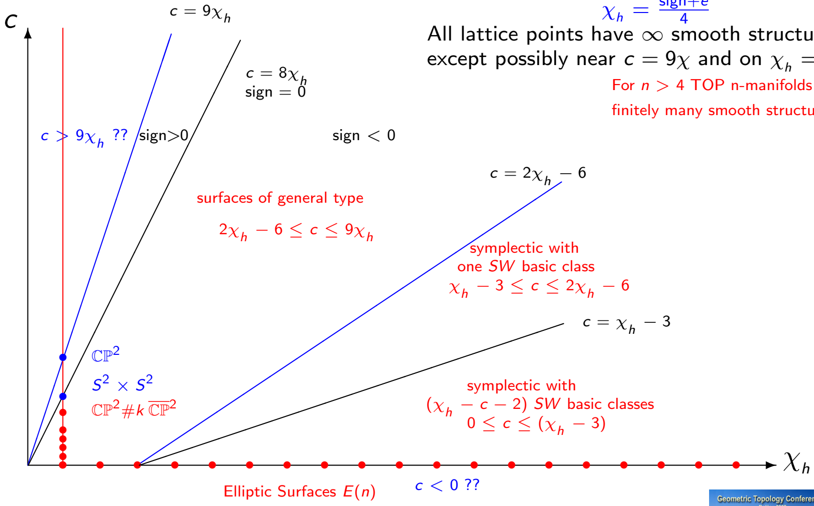
All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



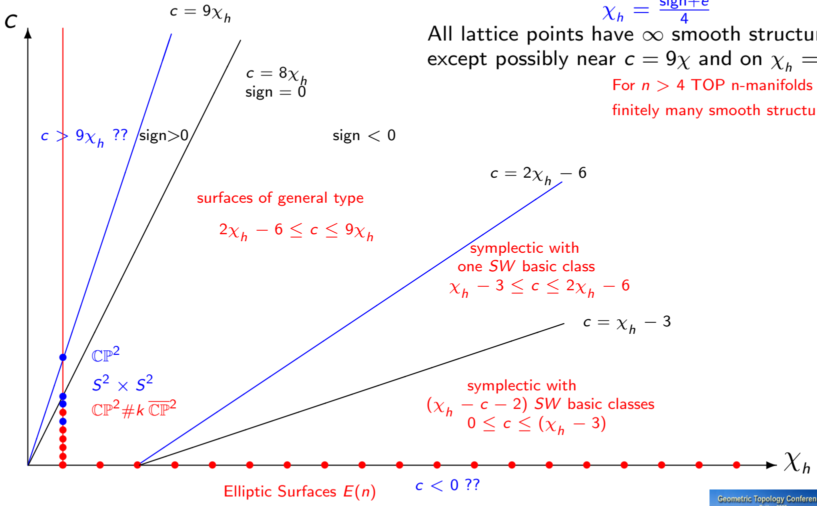
Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

All manifolds minimal
$c = 3\text{sign} + 2e$
$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Oriented minimal 4-manifolds with $\mathcal{SW} \neq 0$

Geography

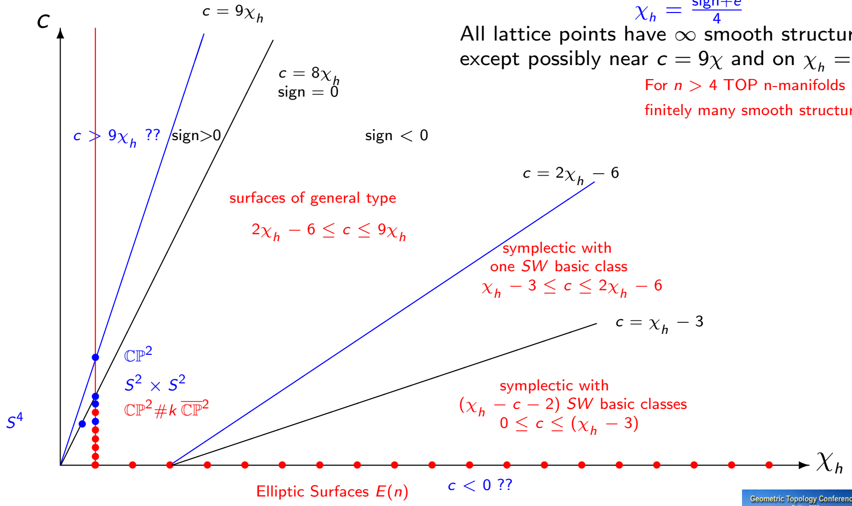
All manifolds minimal

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Construction techniques

We need techniques to attack wild conjectures!

- Surgery/ Log transforms
- Knot surgery
- Fiber sums (including fiber to section)
- Rational blowdown

Construction techniques

We need techniques to attack wild conjectures!

- Surgery/ Log transforms
- Knot surgery
- Fiber sums (including fiber to section)
- Rational blowdown

Construction techniques

We need techniques to attack wild conjectures!

- Surgery/ Log transforms
- Knot surgery
- Fiber sums (including fiber to section)
- Rational blowdown

Construction techniques

We need techniques to attack wild conjectures!

- Surgery/ Log transforms
- Knot surgery
- Fiber sums (including fiber to section)
- Rational blowdown

Construction techniques

We need techniques to attack wild conjectures!

- Surgery/ Log transforms
- Knot surgery
- Fiber sums (including fiber to section)
- Rational blowdown

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \setminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \setminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

Surgery/ Log transform

T: square 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \setminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T)$

$\varphi(\mathrm{pt} \times \partial D^2) =$ surgery curve

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \setminus N_T))$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$

Write $X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$

Due to Morgan, Mrowka, and Szabo.

The Morgan, Mrowka, Szabo Formula

$$\sum_i SW_{X_{T(p,q,r)}}(k+2i[T_{(p,q,r)}]) = p\sum_i SW_{X_{T(1,0,0)}}(k'+2i[T_{(1,0,0)}])$$
$$+q\sum_i SW_{X_{T(0,1,0)}}(k''+2i[T_{(0,1,0)}]) + r\sum_i SW_X(k'''+2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccccccc}
H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) & & k & \to & \bar{k} \\
 & & \downarrow\cong & & & & \downarrow \\
 & & H_2(X \setminus N_T, \partial) & & & & \hat{k} = \hat{k}' \\
 & & \uparrow\cong & & & & \uparrow \\
H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}}) & & k' & \to & \bar{k}'
\end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Important to understand situations when sums collapse to single summand.

The Morgan, Mrowka, Szabo Formula

$$\sum_i \mathrm{SW}_{X_{T(p,q,r)}}(k + 2i[T_{(p,q,r)}]) = p \sum_i \mathrm{SW}_{X_{T(1,0,0)}}(k' + 2i[T_{(1,0,0)}])$$
$$+ q \sum_i \mathrm{SW}_{X_{T(0,1,0)}}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \mathrm{SW}_X(k''' + 2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccc}
H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) \\
 & & \downarrow\cong \\
 & & H_2(X \setminus N_T, \partial) \\
 & & \uparrow\cong \\
H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}})
\end{array}
\qquad
\begin{array}{ccc}
k & \to & \bar{k} \\
 & & \downarrow \\
 & & \hat{k} = \hat{k}' \\
 & & \uparrow \\
k' & \to & \bar{k}'
\end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Important to understand situations when sums collapse to single summand.

The Morgan, Mrowka, Szabo Formula

$$\sum_i \mathrm{SW}_{X_{T(p,q,r)}}(k+2i[T_{(p,q,r)}]) = p\sum_i \mathrm{SW}_{X_{T(1,0,0)}}(k'+2i[T_{(1,0,0)}])$$
$$+q\sum_i \mathrm{SW}_{X_{T(0,1,0)}}(k''+2i[T_{(0,1,0)}]) + r\sum_i \mathrm{SW}_X(k'''+2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccccccc}
H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) & & k & \to & \bar{k} \\
 & & \downarrow\cong & & & & \downarrow \\
 & & H_2(X \setminus N_T, \partial) & & & & \hat{k} = \hat{k}' \\
 & & \uparrow\cong & & & & \uparrow \\
H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}}) & & k' & \to & \bar{k}'
\end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Important to understand situations when sums collapse to single summand.

The Morgan, Mrowka, Szabo Formula

$$\sum_i \mathrm{SW}_{X_{T(p,q,r)}}(k + 2i[T_{(p,q,r)}]) = p \sum_i \mathrm{SW}_{X_{T(1,0,0)}}(k' + 2i[T_{(1,0,0)}])$$
$$+q \sum_i \mathrm{SW}_{X_{T(0,1,0)}}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \mathrm{SW}_X(k''' + 2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccccccc}
H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) & \qquad & k & \to & \bar{k} \\
 & & \downarrow\cong & & & & \downarrow \\
 & & H_2(X \smallsetminus N_T, \partial) & & & & \hat{k} = \hat{k}' \\
 & & \uparrow\cong & & & & \uparrow \\
H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}}) & & k' & \to & \bar{k}'
\end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Important to understand situations when sums collapse to single summand.

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.

$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.

(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)

Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X

α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$

$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \setminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \setminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \setminus N_{T_{(1,0,r)}} = X \setminus N_T$.

b. T nullhomologous, α bounds in $X \setminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)
Let α' = surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \setminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \setminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \setminus N_{T_{(1,0,r)}} = X \setminus N_T$.

b. T nullhomologous, α bounds in $X \setminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.

$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.

(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)

Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X

α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$

$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Surgery

Reducing to one summand

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.

$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.

(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)

Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X

α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$

$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$X_K = (X \smallsetminus N_T) \cup (S^1 \times (S^3 \smallsetminus N_K))$

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$X_K = (X \setminus N_T) \cup (S^1 \times (S^3 \setminus N_K))$

Facts about knot surgery

- If X and $X \setminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$X_K = (X \setminus N_T) \cup (S^1 \times (S^3 \setminus N_K))$

Facts about knot surgery

- If X and $X \setminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$X_K = (X \smallsetminus N_T) \cup (S^1 \times (S^3 \smallsetminus N_K))$

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$$X_K = (X \smallsetminus N_T) \cup (S^1 \times (S^3 \smallsetminus N_K))$$

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$$X_K = (X \smallsetminus N_T) \cup (S^1 \times (S^3 \smallsetminus N_K))$$

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$X_K = (X \setminus N_T) \cup (S^1 \times (S^3 \setminus N_K))$

Facts about knot surgery

- If X and $X \setminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

Definition

$$X_K = (X \smallsetminus N_T) \cup (S^1 \times (S^3 \smallsetminus N_K))$$

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusion

- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Techniques Using Rational Blowdown

Blowing down

Techniques Using Rational Blowdown

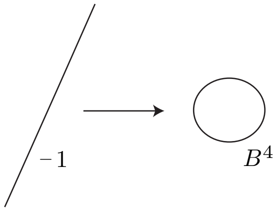
Blowing down

Sphere $E \subset X$ with $E \cdot E = -1$.
$N_E \cong -(\text{Nbd } \mathbb{CP}^1 \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \setminus \mathbb{CP}^1 \cong B^4$.

Techniques Using Rational Blowdown

Blowing down

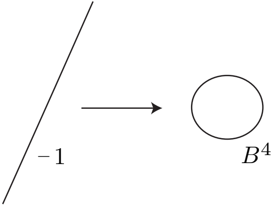
Sphere $E \subset X$ with $E \cdot E = -1$.
$N_E \cong -(\text{Nbd } \mathbb{CP}^1 \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \smallsetminus \mathbb{CP}^1 \cong B^4$.
Trade N_E for B^4.
Simple formula for SW



Techniques Using Rational Blowdown

Blowing down

Sphere $E \subset X$ with $E \cdot E = -1$.
$N_E \cong -(\text{Nbd } \mathbb{CP}^1 \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \smallsetminus \mathbb{CP}^1 \cong B^4$.
Trade N_E for B^4.
Simple formula for SW

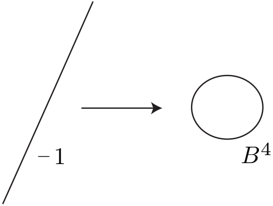


Blowing down a -4-sphere

Techniques Using Rational Blowdown

Blowing down

Sphere $E \subset X$ with $E \cdot E = -1$.
$N_E \cong -(\text{Nbd } \mathbb{CP}^1 \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \smallsetminus \mathbb{CP}^1 \cong B^4$.
Trade N_E for B^4.
Simple formula for SW



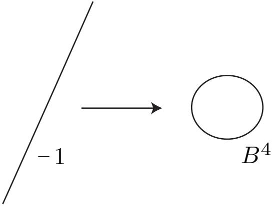
Blowing down a -4-sphere

Sphere $S \subset X$ with $S \cdot S = -4$.
$N_S \cong -(\text{Nbd conic } \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \smallsetminus \{\text{conic}\} \cong \text{Nbd}(\mathbb{RP}^2) = B_2$.

Techniques Using Rational Blowdown

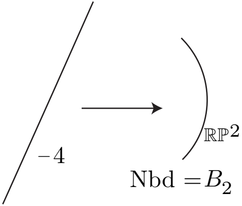
Blowing down

Sphere $E \subset X$ with $E \cdot E = -1$.
$N_E \cong -(\text{Nbd } \mathbb{CP}^1 \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \setminus \mathbb{CP}^1 \cong B^4$.
Trade N_E for B^4.
Simple formula for SW



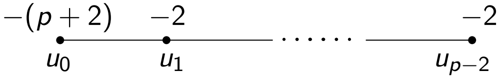
Blowing down a -4-sphere

Sphere $S \subset X$ with $S \cdot S = -4$.
$N_S \cong -(\text{Nbd conic } \subset \mathbb{CP}^2)$.
$\mathbb{CP}^2 \setminus \{\text{conic}\} \cong \text{Nbd}(\mathbb{RP}^2) = B_2$.
Trade N_S for B_2.
Simple formula for SW



Rational blowdown in general

Suppose X contains a configuration C_p of 2-spheres



$\partial C_p = L(p^2, 1-p)$,

which bounds a rational homology ball B_p.

Rationally blowdown X by replacing C_p with B_p.

Simple formula for change in SW-invariant.

Process decreases b^- by $p-1$, leaves b^+ unchanged.

Rational blowdown in general

Suppose X contains a configuration C_p of 2-spheres

$$\underset{u_0}{\overset{-(p+2)}{\bullet}} \text{———} \underset{u_1}{\overset{-2}{\bullet}} \text{———} \cdots\cdots \text{———} \underset{u_{p-2}}{\overset{-2}{\bullet}}$$

$\partial C_p = L(p^2, 1-p)$,

which bounds a rational homology ball B_p.

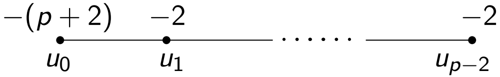
Rationally blowdown X by replacing C_p with B_p.

Simple formula for change in SW-invariant.

Process decreases b^- by $p-1$, leaves b^+ unchanged.

Rational blowdown in general

Suppose X contains a configuration C_p of 2-spheres



$\partial C_p = L(p^2, 1-p)$,

which bounds a rational homology ball B_p.

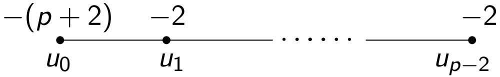
Rationally blowdown X by replacing C_p with B_p.

Simple formula for change in SW-invariant.

Process decreases b^- by $p-1$, leaves b^+ unchanged.

Rational blowdown in general

Suppose X contains a configuration C_p of 2-spheres



$\partial C_p = L(p^2, 1-p)$,

which bounds a rational homology ball B_p.

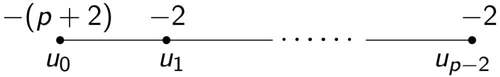
Rationally blowdown X by replacing C_p with B_p.

Simple formula for change in SW-invariant.

Process decreases b^- by $p-1$, leaves b^+ unchanged.

Rational blowdown in general

Suppose X contains a configuration C_p of 2-spheres



$\partial C_p = L(p^2, 1-p)$,

which bounds a rational homology ball B_p.

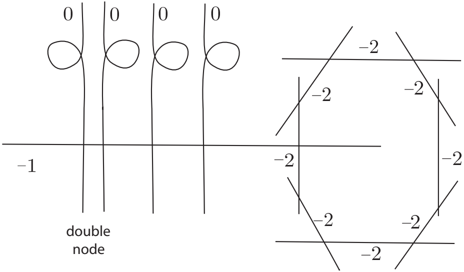
Rationally blowdown X by replacing C_p with B_p.

Simple formula for change in SW-invariant.

Process decreases b^- by $p-1$, leaves b^+ unchanged.

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

Start with elliptic surface $E(1)$.

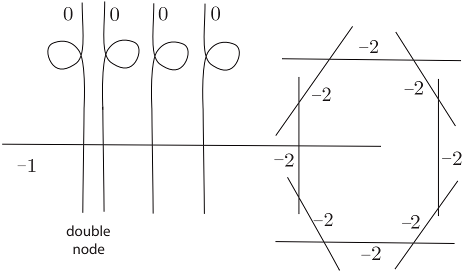


Perform knot surgery in the double node nbd using twist knot

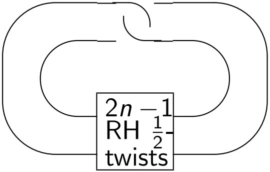
K = twist knot

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

Start with elliptic surface $E(1)$.



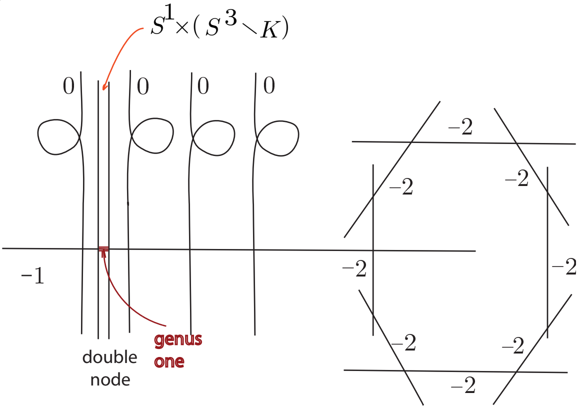
Perform knot surgery in the double node nbd using twist knot



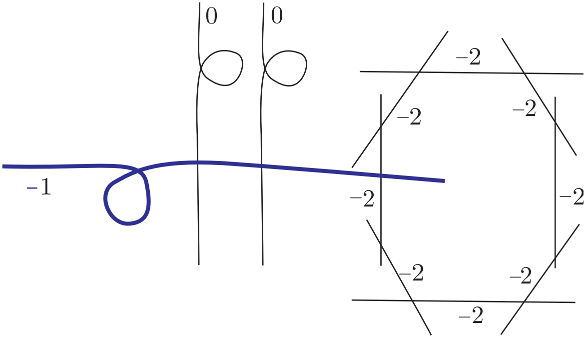
K = twist knot

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

In $E(1)_K$:



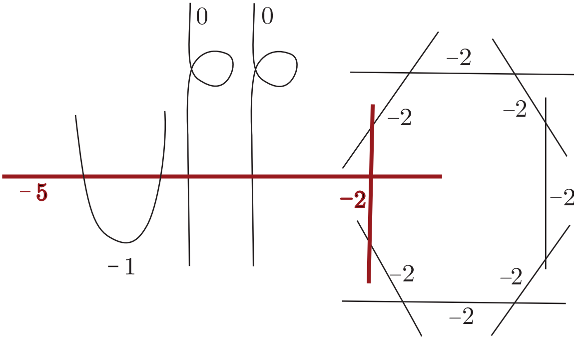
The double node trick trades the 'genus one pseudosection' for an immersed sphere:



Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 8$

Blow up the double point of the pseudosection:



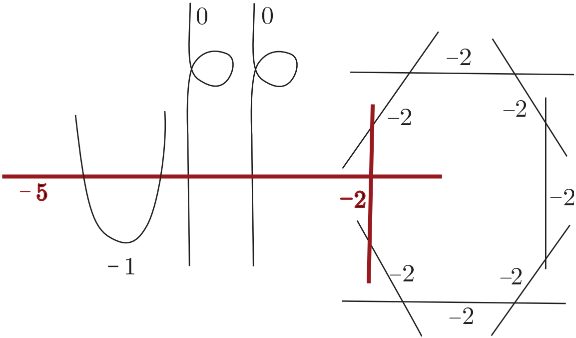
$C_3 \subset E(1)_K \# \overline{\mathbb{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ by varying $K =$ twist knot

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 8$

Blow up the double point of the pseudosection:



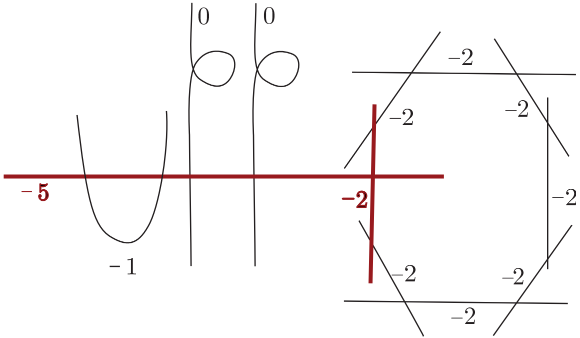
$C_3 \subset E(1)_K \# \overline{\mathbb{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ by varying K = twist knot

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 8$

Blow up the double point of the pseudosection:



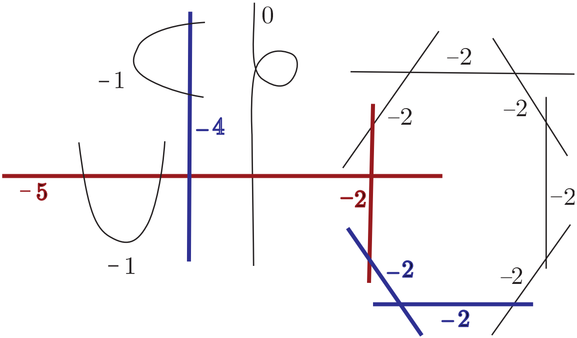
$C_3 \subset E(1)_K \# \overline{\mathbb{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ by varying $K =$ twist knot

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 7$

Blow up the double point of one of the nodal fibers:



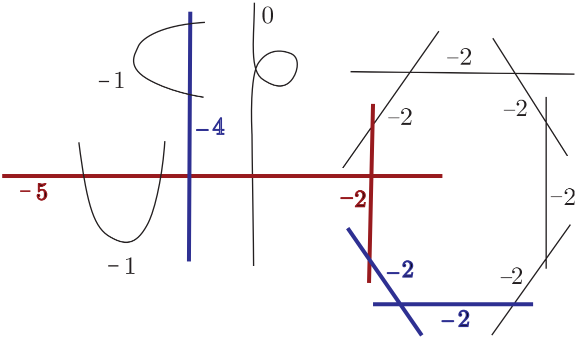
Smooth intersection of “-4” with “−5”

Get a sphere of self intersection −7 in $E(1)_K \# 2\,\overline{\mathbb{CP}}^2$

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 7$

Blow up the double point of one of the nodal fibers:



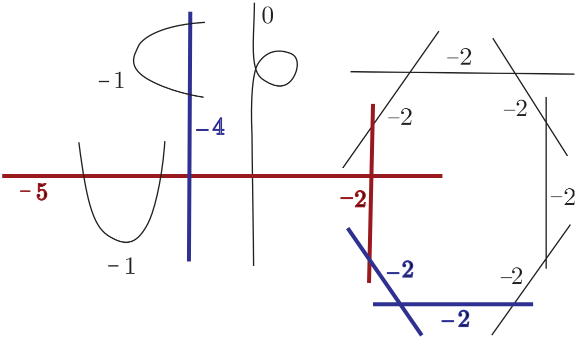
Smooth intersection of "-4" with "-5"

Get a sphere of self intersection -7 in $E(1)_K \# 2\overline{\mathbb{CP}}^2$

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 7$

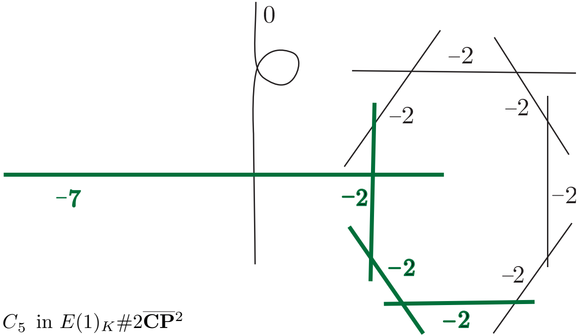
Blow up the double point of one of the nodal fibers:



Smooth intersection of "-4" with "-5"

Get a sphere of self intersection -7 in $E(1)_K \# 2\,\overline{\mathbb{CP}}^2$

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$ $b^- = 7$(cont.)



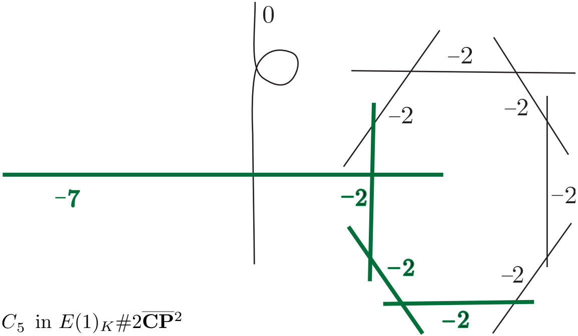
C_5 in $E(1)_K\#2\overline{\mathbf{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ by varying K = twist knot

Similar techniques ∞ families homeo to $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 7$(cont.)



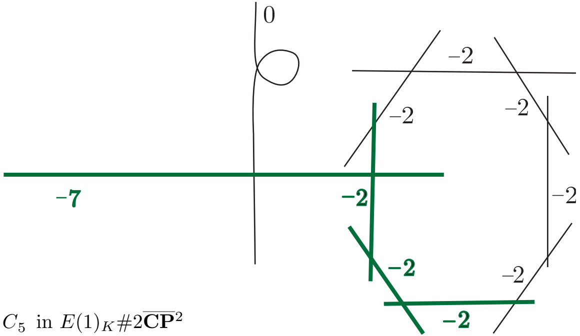
C_5 in $E(1)_K\#2\overline{\mathbf{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$ by varying K = twist knot

Similar techniques ∞ families homeo to $\mathbb{CP}^2\#6\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$

Some exotic $\pi_1 = 0$ manifolds with $b^+ = 1$

$b^- = 7$(cont.)



C_5 in $E(1)_K \# 2\overline{\mathbf{CP}}^2$

Rationally blow down to get ∞ family homeo to $\mathbb{CP}^2 \# 7\,\overline{\mathbb{CP}}^2$ by varying K = twist knot

Similar techniques ∞ families homeo to $\mathbb{CP}^2 \# 6\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p, 0, 1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p, 0, 1)) = 0$, and if $\pi_1(X_T(p, 0, 1)) = 0$, then $X_T(p, 0, 1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p, 0, 1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering

Idea behind reverse engineering

Let T be a nullhomologous torus in X.

Morgan, Mrowka, Szabo formula says (more-or-less) that

$$\mathcal{SW}_{X_T(p,0,1)} = p\,\mathcal{SW}_{X_T(1,0,0)} + \mathcal{SW}_X$$

So if $\mathcal{SW}_{X_T(1,0,0)} \neq 0$ then we get an infinite family.

Suppose X is simply connected. For $(p,0,1)$ - surgery on T with nullhomologous α:

- $H_1(X_T(p,0,1)) = 0$, and if $\pi_1(X_T(p,0,1)) = 0$, then $X_T(p,0,1)$ is homeomorphic to X.
- The new core torus $T_{(p,0,1)}$ is nullhomologous in $X_T(p,0,1)$
- If $T_{(1,0,0)}$ has a 'dual' torus T' of square 0 such that $T' \cdot T_{(1,0,0)} \neq 0$, then the corresponding sum in the M-M-Sz Formula collapses to one term.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

Reverse Engineering in Practice

Find a model manifold

Suppose we wish to construct a family of manifolds homeomorphic to the simply connected manifold Q.

Surgery on a torus changes neither the euler number nor signature.

1. Start with a model manifold M which has the same e and sign as Q, but with $b_1 > 0$.
2. Want disjoint homologically primitive tori in M, each of which contains a generator of $H_1(M)$, and such that surgeries on these tori reduce $b_1(M)$ to 0.
3. Also want each such torus to have a dual as above.
4. Want to order the surgeries so that the next-to-last manifold (with $b_1 = 1$) has $\mathcal{SW} \neq 0$.

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p,0,1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p,0,1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p,0,1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p, 0, 1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

A Simple Example

Suppose Y is a 4-manifold with

- $b_1 = 1$
- Contains a homologically primitive square 0 torus T with a loop α on T representing a generator of $H_1(Y;\mathbb{Z})$.
- Contains square 0 torus T' with $T' \cdot T \neq 0$
- $\mathcal{SW}_Y \neq 0$

Perform a surgery on T which kills α homologically — get X.

- $b_1(X) = 0$, $H_2(Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus$ hyperbolic pair
- $\Lambda =$ core torus of surgery is nullhomologous in X.
- $\exists$ loop λ on Λ so that certain (0-) surgery gives back Y.

Extend λ to basis of $H_1(\Lambda)$ and do $(p,0,1)$-surgery to get X_p.

$\mathcal{SW}_{X_p} = p\,\mathcal{SW}_Y + \mathcal{SW}_X$

- Infinite family

Families

- **The $\mathcal{SW}$ condition**

  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.

- **Simple connectivity**

  Easier in some cases than others

- **Infinite families**

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- **The $\mathcal{SW}$ condition**
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- **Simple connectivity**
  Easier in some cases than others
- **Infinite families**

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition

  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.

- Simple connectivity

  Easier in some cases than others

- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- The $\mathcal{SW}$ condition
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- Simple connectivity
  Easier in some cases than others
- Infinite families

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Families

- **The $\mathcal{SW}$ condition**
  If M is symplectic and the tori chosen for surgery are Lagrangian and we do $(p, 0, \pm 1)$ surgery with respect to the Lagrangian framing of of one of these tori then the resultant manifold will again be symplectic and so it has $\mathcal{SW} \neq 0$.
- **Simple connectivity**
  Easier in some cases than others
- **Infinite families**

  Above surgery process ends with

1. $H_1 = 0$ (simply connected, if lucky) manifold X
2. Nullhomologous torus $\Lambda \subset X$
3. Loop λ on Λ with nullhomologous pushoff so that corresponding surgery gives X', the next-to-last manifold with $\mathcal{SW}_{X'} \neq 0$. ($X' = X_\Lambda(1, 0, 0)$)

- Infinite family

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

Model Manifold $= Sym^2(\Sigma_3)$

Has the same e and sign as $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Has $\pi_1 = H_1(\Sigma_3)$ (so $b_1 = 6$)

Is symplectic and has plenty of disjoint Lagrangian tori with 'duals'.

- Six surgeries give a simply connected symplectic X whose canonical class pairs positively with the symplectic form.
- Not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ since each symplectic form on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ pairs negatively with its canonical class. Li-Liu
- Get infinite family of distinct manifolds all homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (joint with Ron Stern and Doug Park)

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\,\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and $S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and $S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$$(S^1 \times M_K \# 2\,\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$$(S^1 \times M_K \# 2\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$$(S^1 \times M_K \# 2\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\,\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and $S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\,\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Comparison with other fake $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s

The Baldridge-Kirk Example

The model manifold is: $(T^4 \# \overline{\mathbb{CP}}^2) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_2} \Sigma_2 \times T^2$

$T^4 \# \overline{\mathbb{CP}}^2 = Sym^2(\Sigma_2)$

Conjecture

$$Sym^2(\Sigma_{g+1}) \cong Sym^2(\Sigma_g) \# \overline{\mathbb{CP}}^2 \#_{\Sigma_g} \Sigma_g \times T^2$$

Prove by watching Σ_{g+1} degenerate to $\Sigma_g \vee T^2$?

The Akhmedov-Park Example

$(S^1 \times M_K \# 2\,\overline{\mathbb{CP}}^2) \#_{\Sigma_2} (S^1 \times M_K \#_{F=S} S^1 \times M_K)$

$S^1 \times M_K$ is obtained from T^4 by two surgeries, and
$S^1 \times M_K \#_{F=S} S^1 \times M_K$ from $\Sigma_2 \times T^2$ after 4 surgeries.

$\Rightarrow$ Same model manifolds

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. ($\Sigma_3/\mathbb{Z}_3 = T^2$). Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2, 2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger. This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

Other Model Manifolds

How do we find them?

- Library skills — read papers on constructions of algebraic surfaces with small p_g and q
- Construct them yourself!

Some examples

- $T^4 = T^2 \times T^2$. Take double branched cover branched over the $(2,2)$ curve (and desingularize). Get manifold with $e = 8$, sign $= -4$, and $b_1 = 4$ and plenty of Lagrangian tori to surger.

  This model gives manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

- $\mathbb{Z}_3$-action on Σ_3 with 2 fixed points. $(\Sigma_3/\mathbb{Z}_3 = T^2)$. Diagonal action on $\Sigma_3 \times \Sigma_3$ descends to $Sym^2(\Sigma_3)$ with 3 fixed points.

  Get model for $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

A Challenge

In $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ find a nullhomologous torus so that surgeries on it give the known fake examples.

Santeria Surgery