Smooth 4-Manifolds: 2011

Ron Fintushel

Michigan State University

Geography of S.C. Minimal Complex Surfaces

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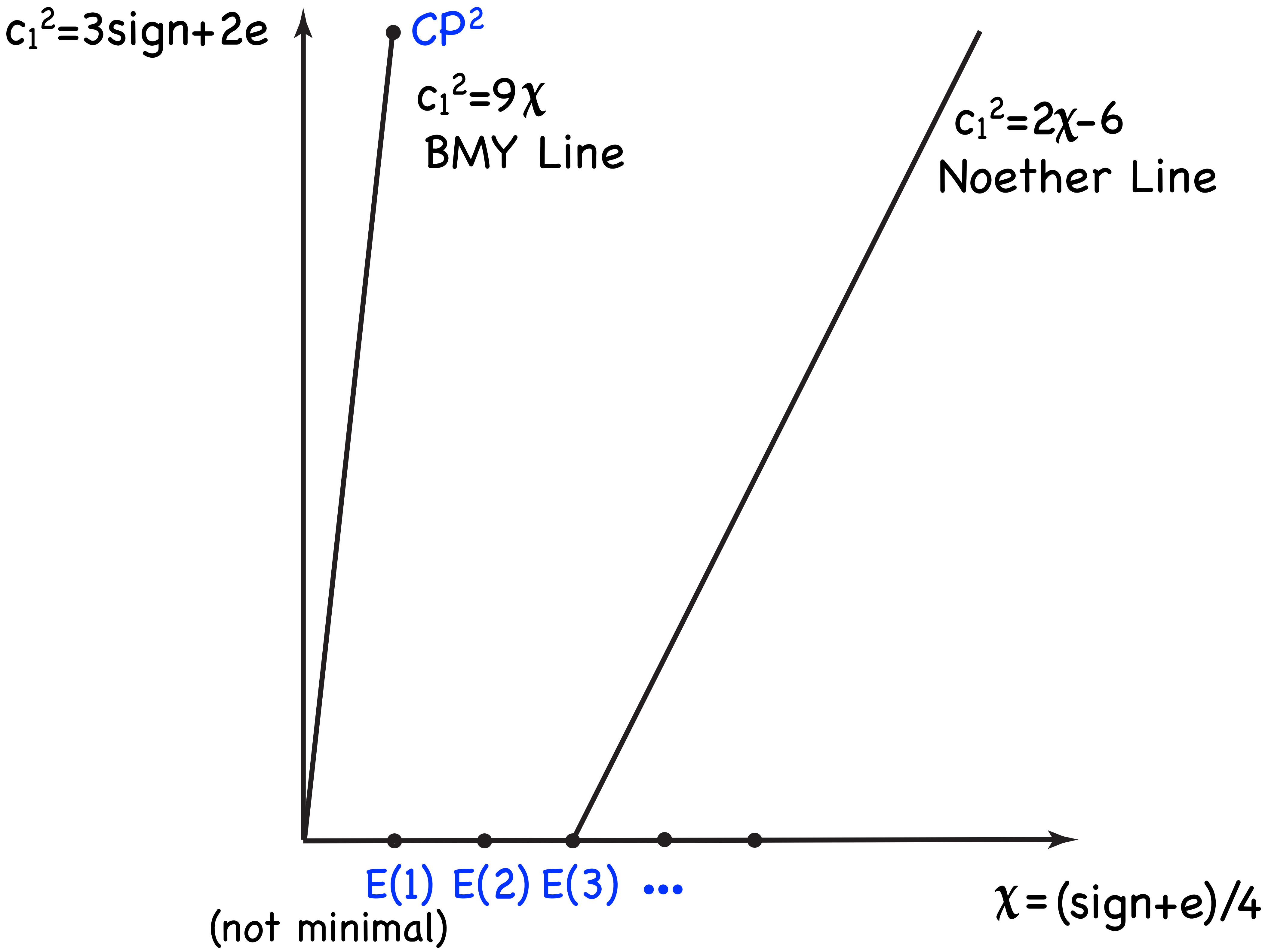

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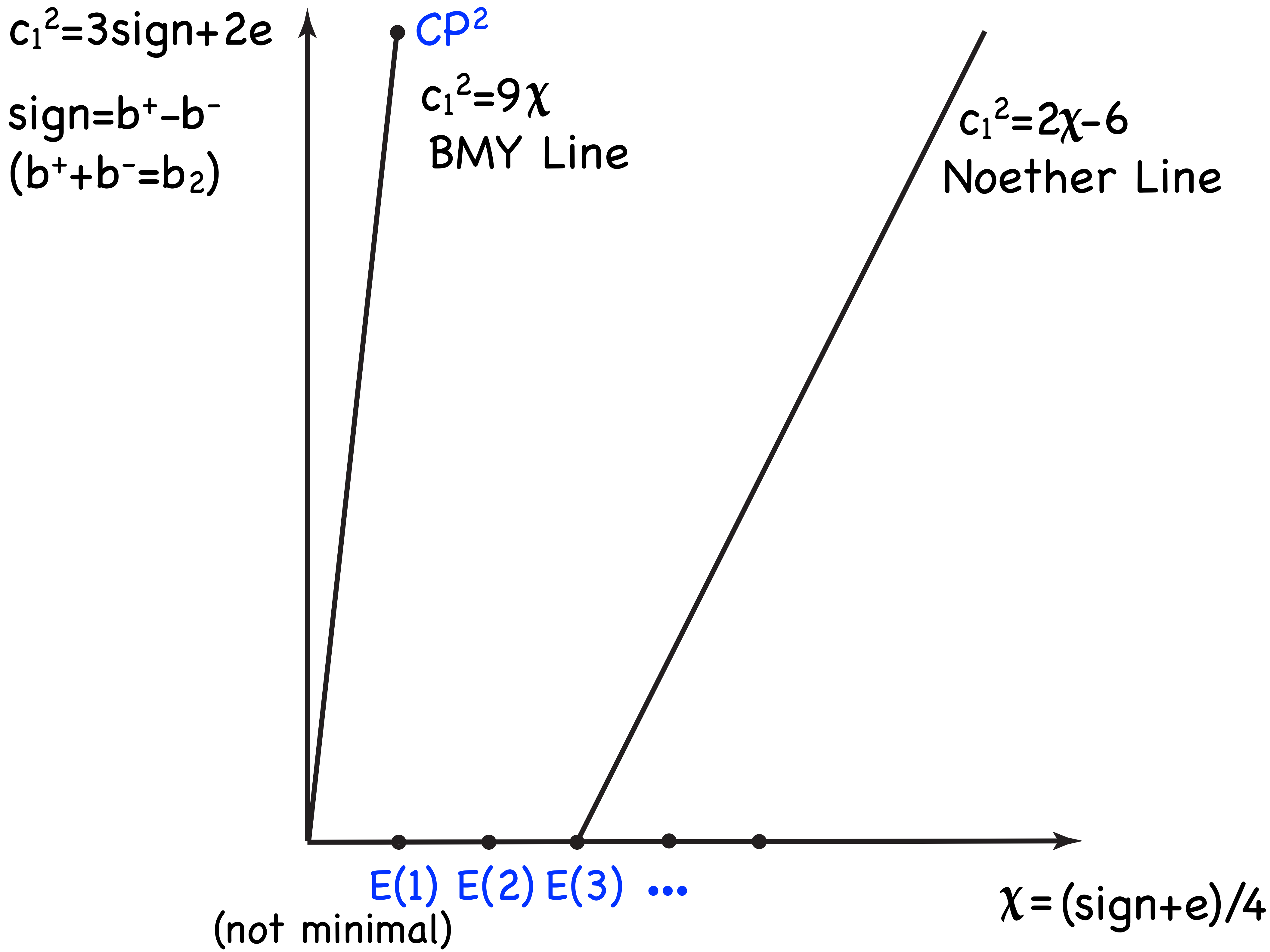

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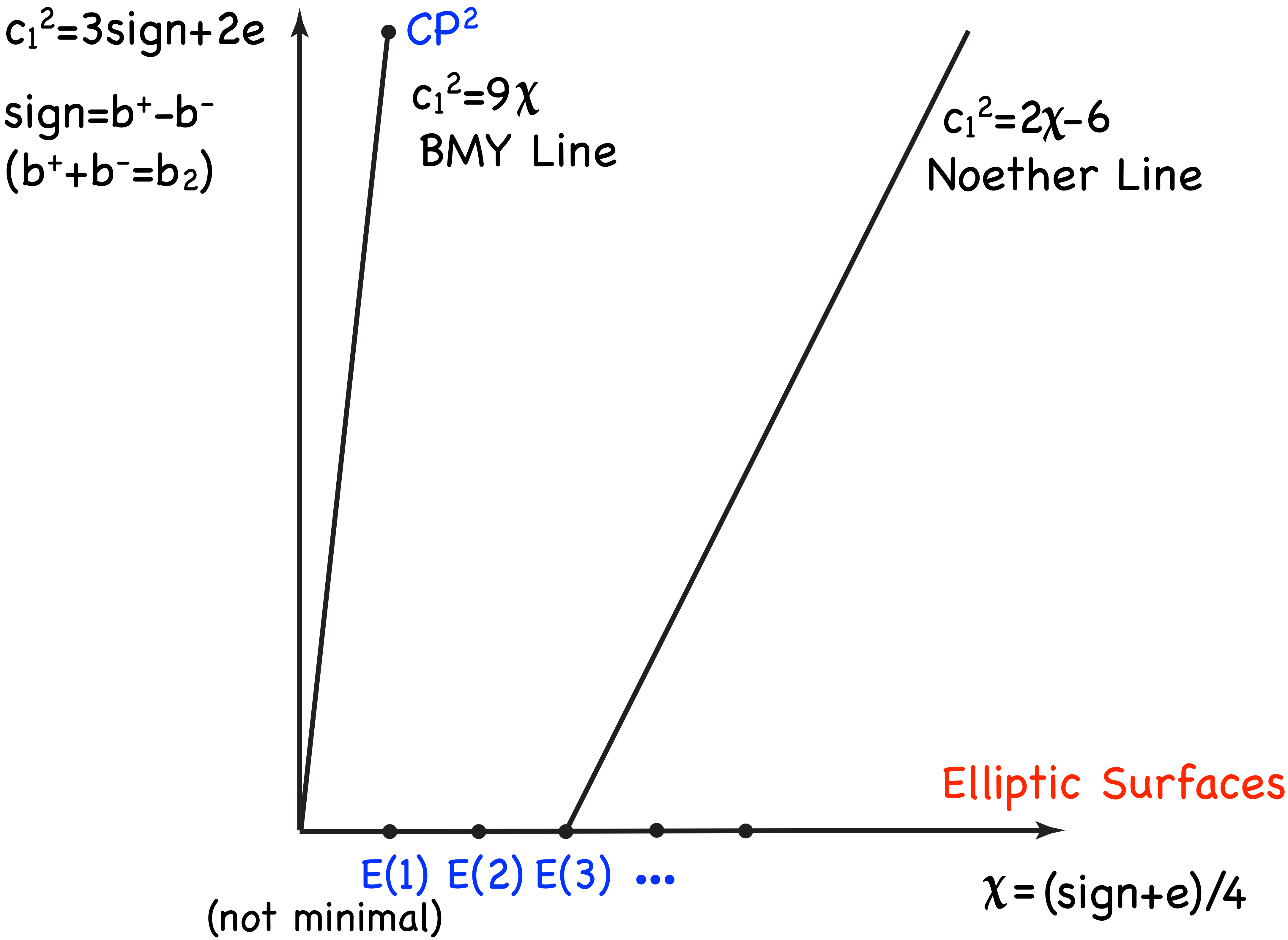

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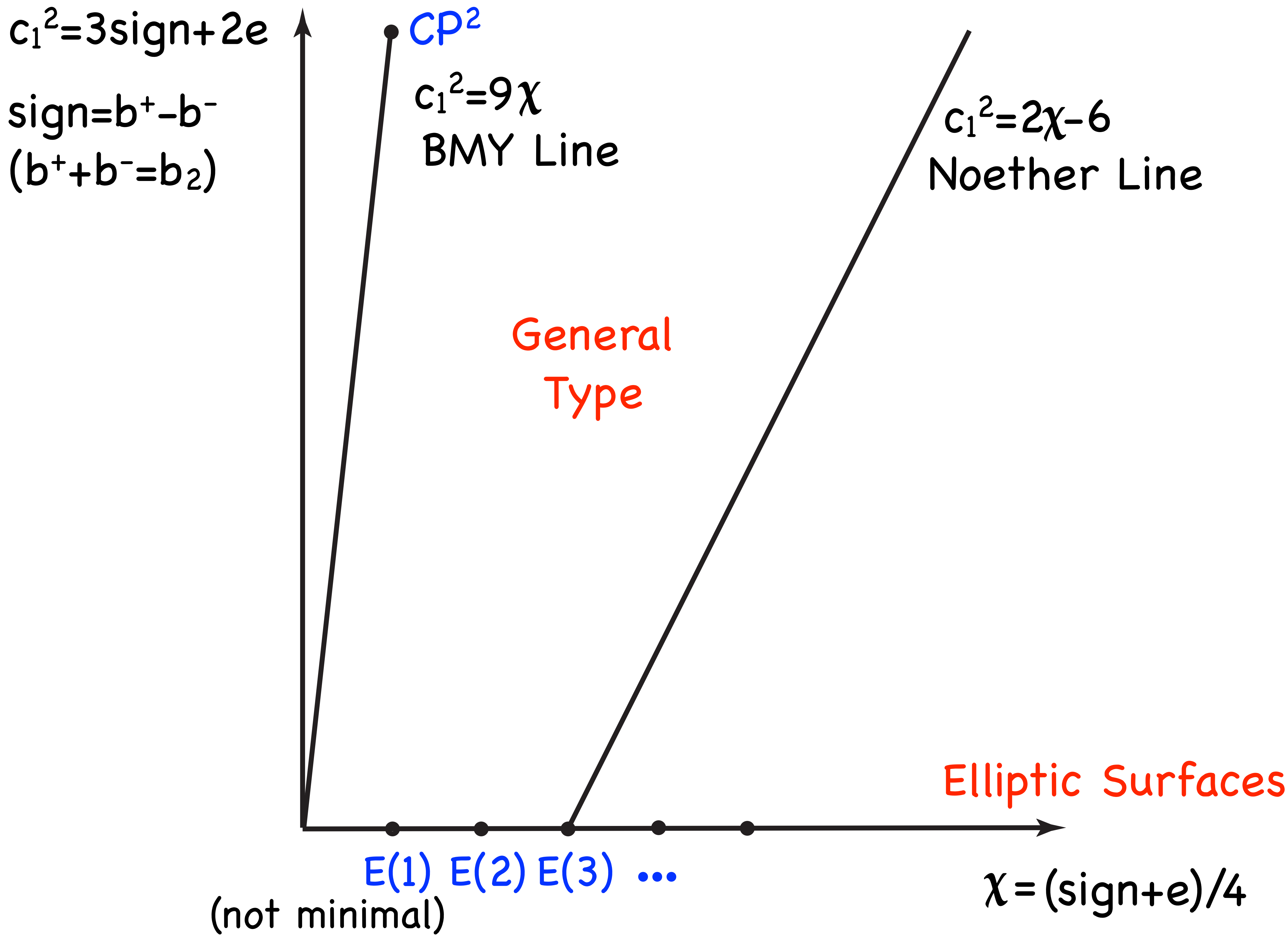

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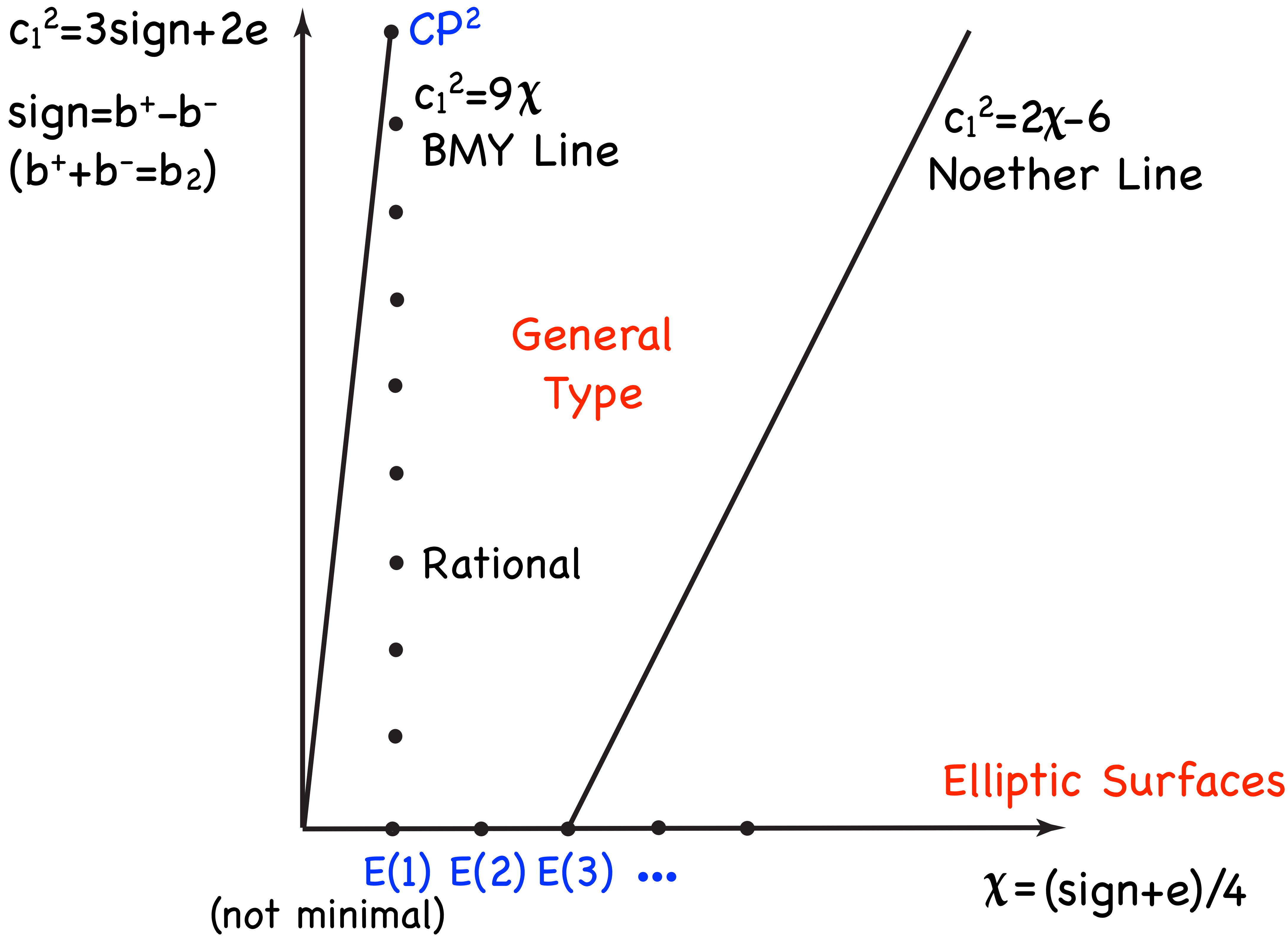

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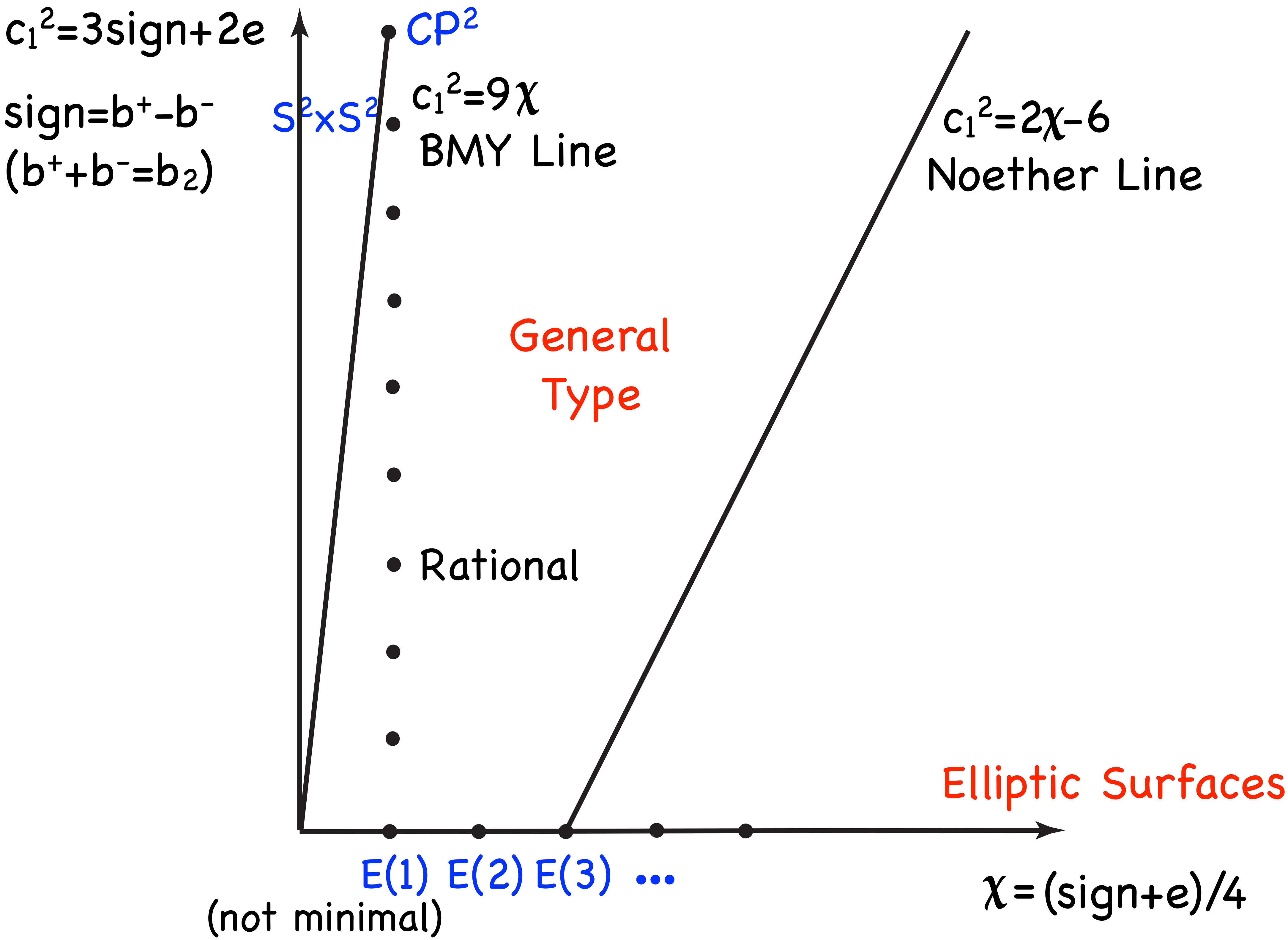

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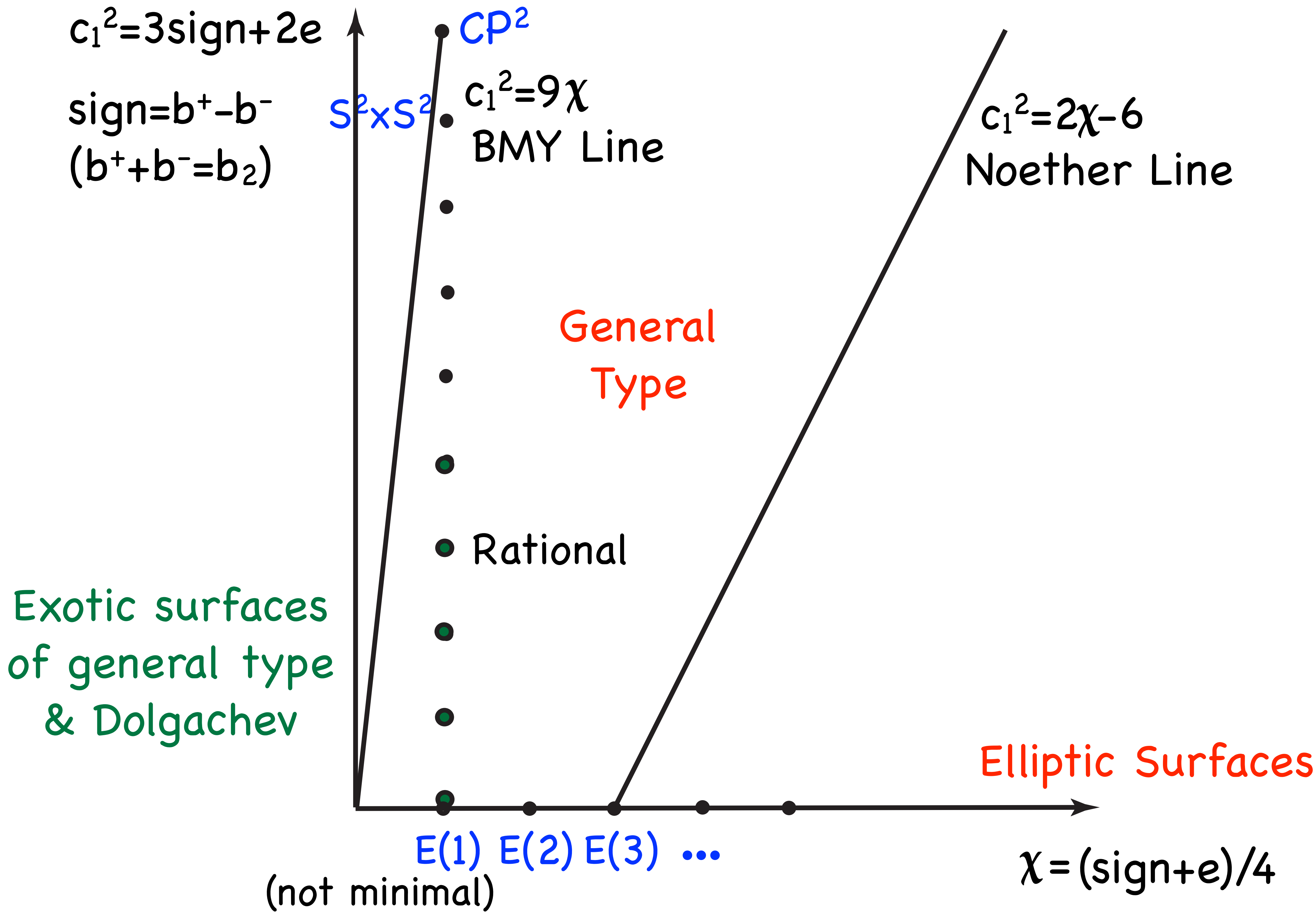

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$\Rightarrow$ Homeo type of s.c. smooth 4-mfd w/ b+=1 determined by Type (even, odd)

rank of H_2

Elliptic Surfaces

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$E(1)=CP^2\#9\overline{CP}^2$

$$\begin{array}{ccc} T^2 & \rightarrow & E(1) \\ & & \downarrow \pi \\ & & CP^1 \end{array}$$

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For elliptic surfaces with cusp fibers the result of a log transform depends only on the multiplicity.
True (up to diffeo) if simply connected.

Exotic Complex Surfaces with $b^+=1$

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$b^-=8$ Barlow surface

homeo to $CP^2\#8\overline{CP}^2$

not diffeo Kotschick, 1989

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For $b^+=1$, minor complications arising from reducible solutions to SW eq'ns for some metrics. Get inv'ts $SW^{\pm}$ and these determine SW.

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Log transform formula: $SW_{X_p}= SW_X.(t^{p-1}+t^{p-3}+...+t^{3-p}+t^{1-p})$

where $t=$ multiple fiber; so $t^p=$fiber

Works for $SW^{\pm}$ when $b^+=1$. Can use to compute SW.

E.g. $SW_{E(1)_{2,3}} = t^{-1}-t$

Knot Surgery

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K: knot in S^3 $\qquad X_K = (X-(T\times D^2)) \cup (S^1 \times (S^3 - N_K))$

glued so that (long. of K) $\leftrightarrow \partial D^2$

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- Works for $SW^{\pm}$ when $b^+=1$

Consequence: K=n-twist knot, $SW_{E(1)_K} = n(t^{-1}-t)$

$\Rightarrow$ no two diffeo, all homeo to E(1)

Rational Blowdown

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Usual blowdown: $S^2 \subset X$, square -1, $N_{S^2} = \overline{CP}^2$-ball

$\partial N_{S^2} = S^3$ Trade N_{S^2} for B^4, get $\overline{X}$ $b^-_{\overline{X}} = b^-_X - 1$

$SW_X = SW_{\overline{X}} \cdot (\varepsilon + \varepsilon^{-1})$

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Blowdown -4 sphere: replace N_{S^2} with $N_{RP^2 \subset CP^2}$

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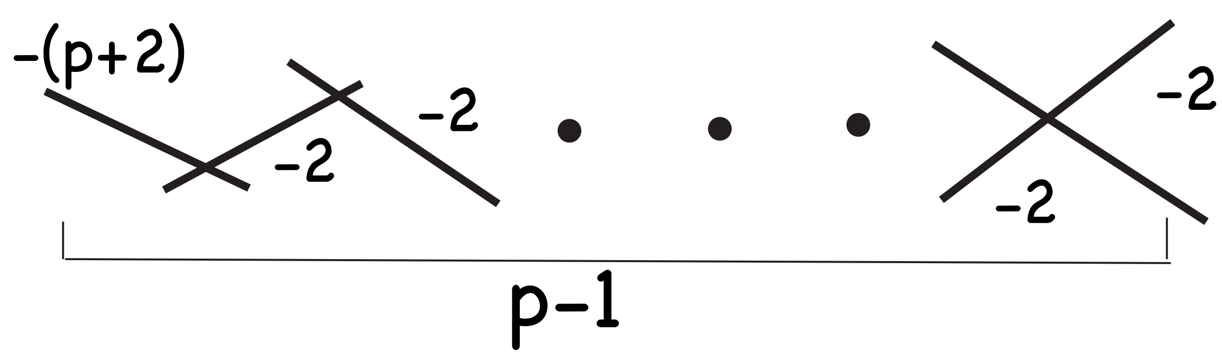

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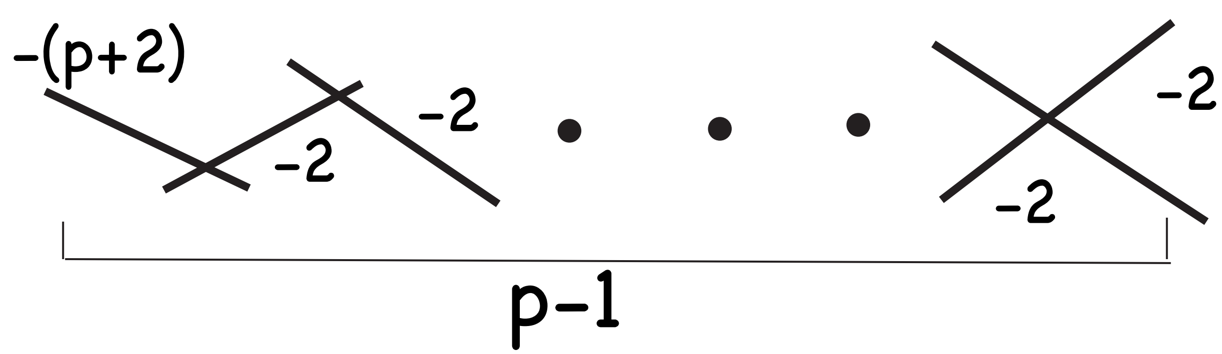
In general, 

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Rational blowdown - remove C_p, glue in B_p. Get $\overline{X}$

Lowers b^- by $p-1$.

Rational Blowdowns, II

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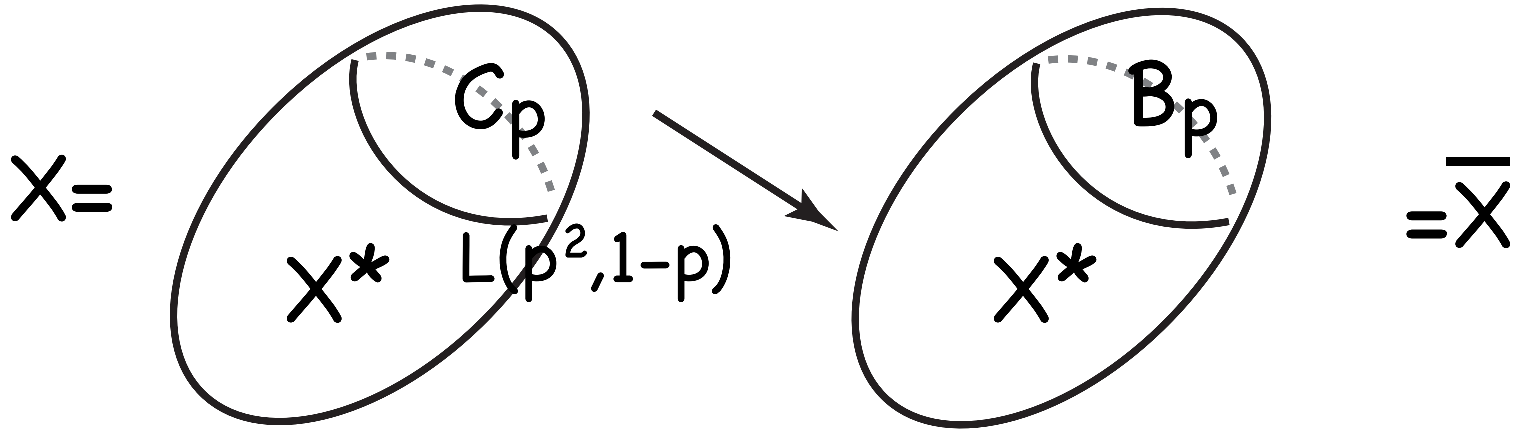

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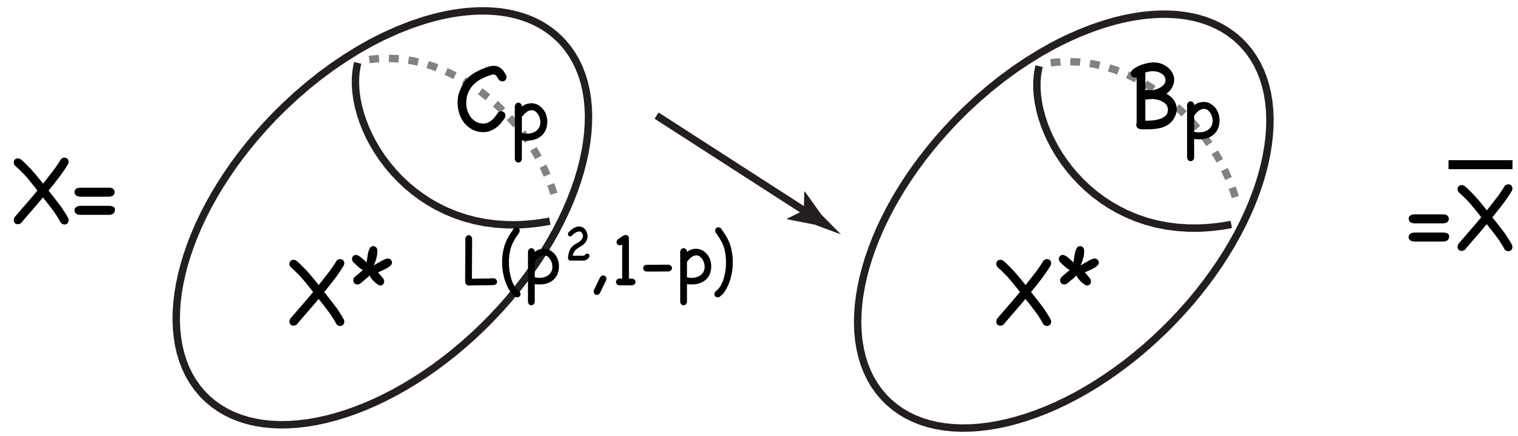

If $\bar{k}$=char homology class in $\bar{X}$, $\exists$ lift k in X $\ni$ PD's agree on X^*

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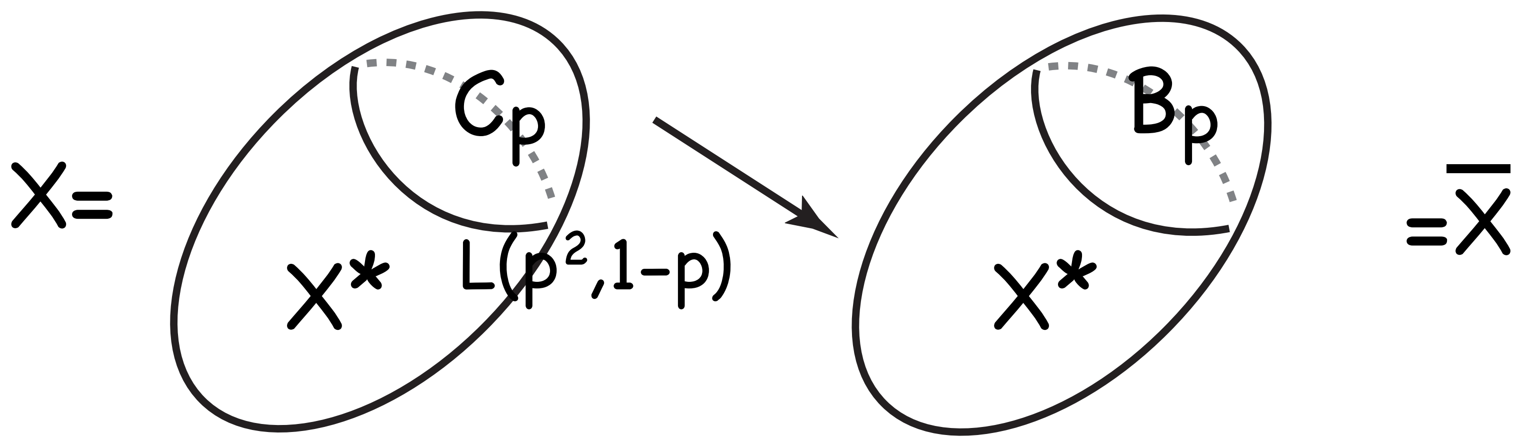

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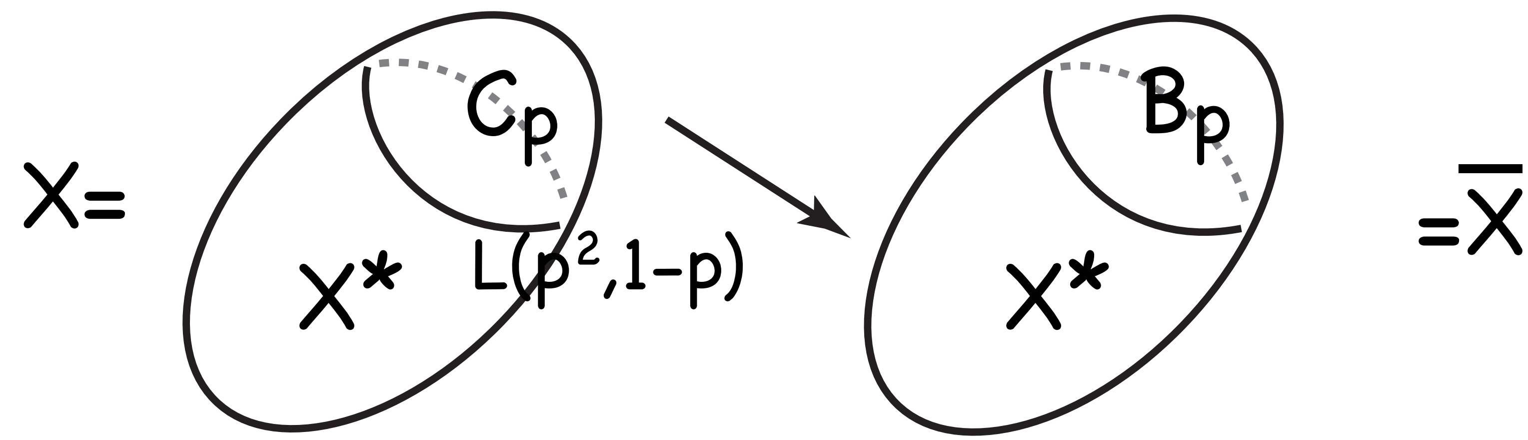

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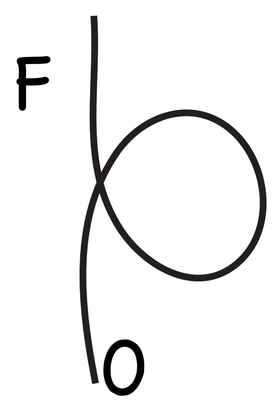

Nodal Fiber
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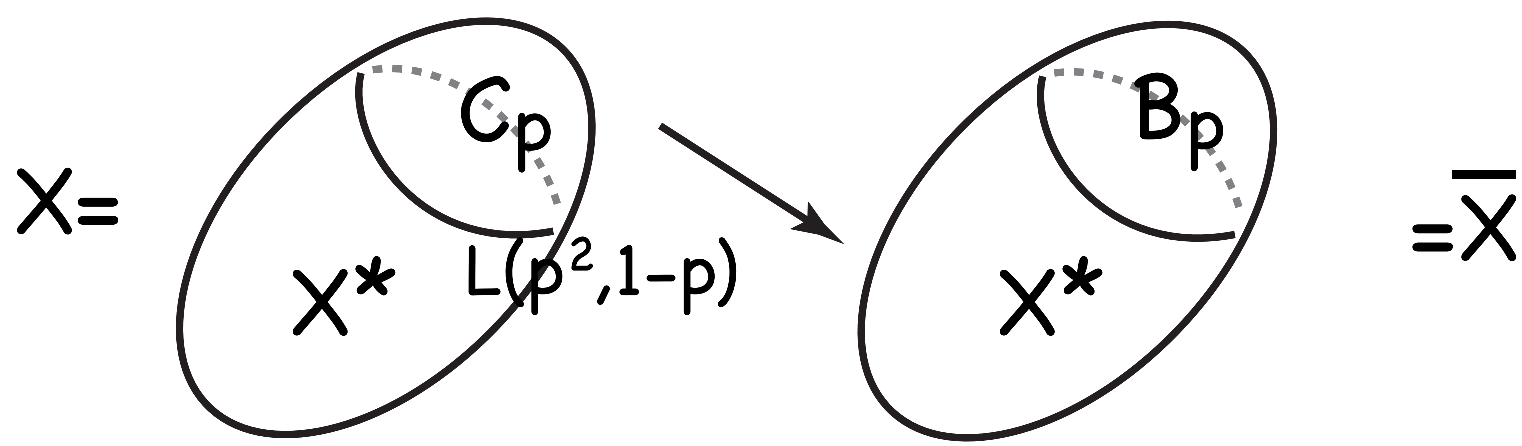

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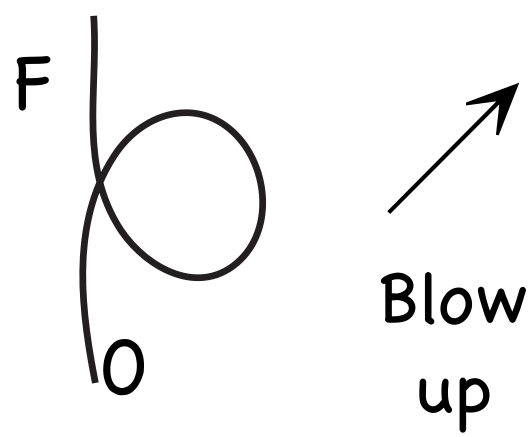

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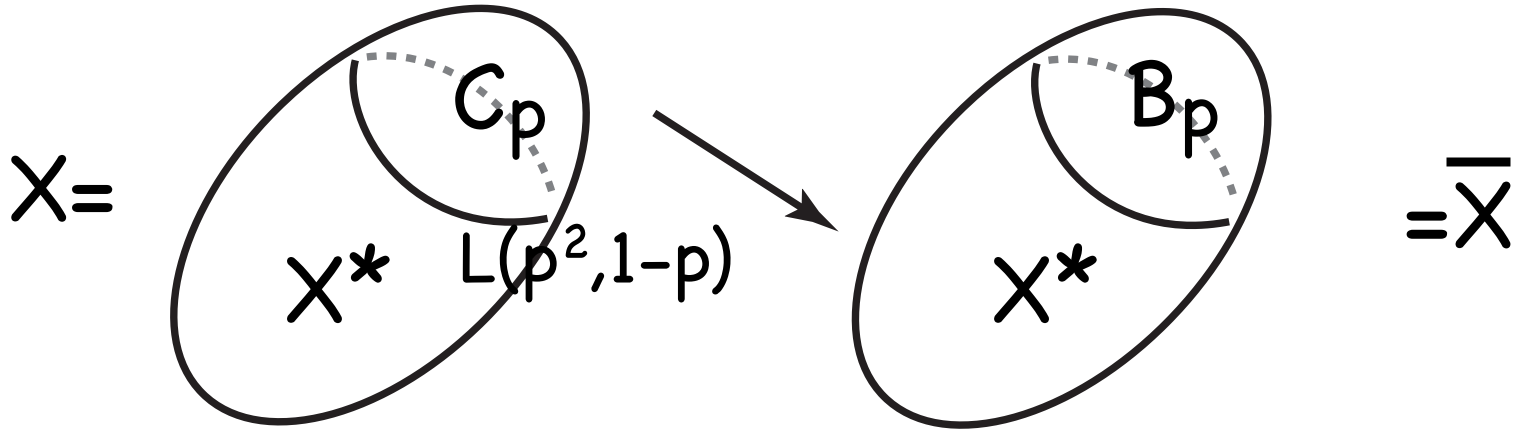

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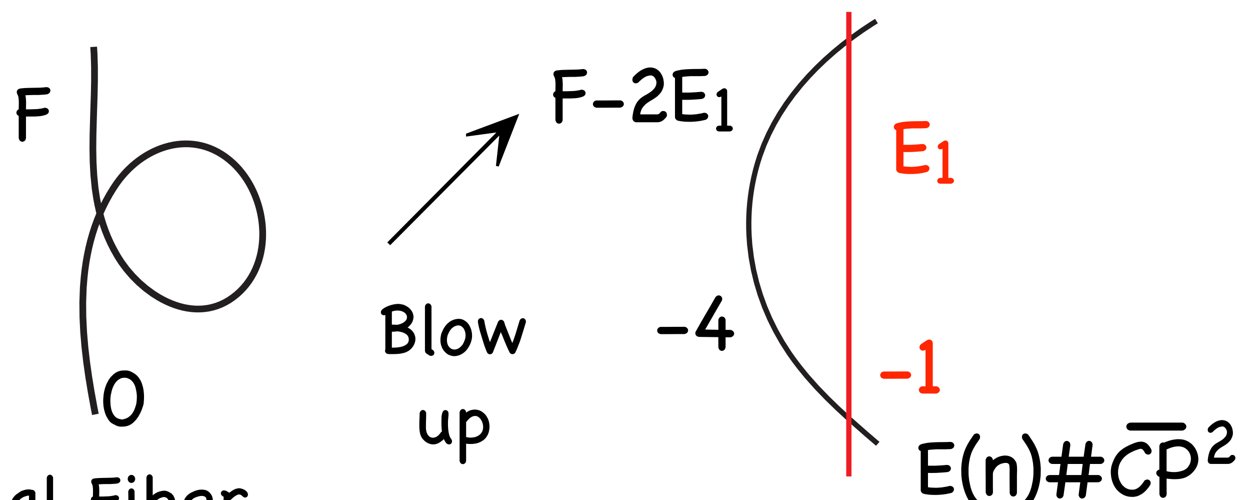

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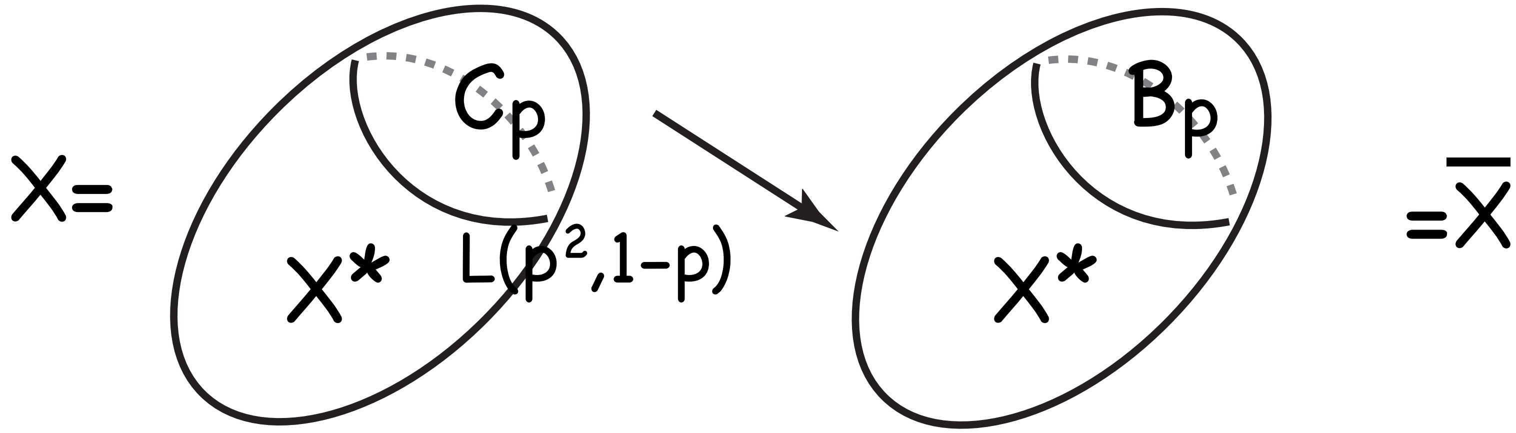

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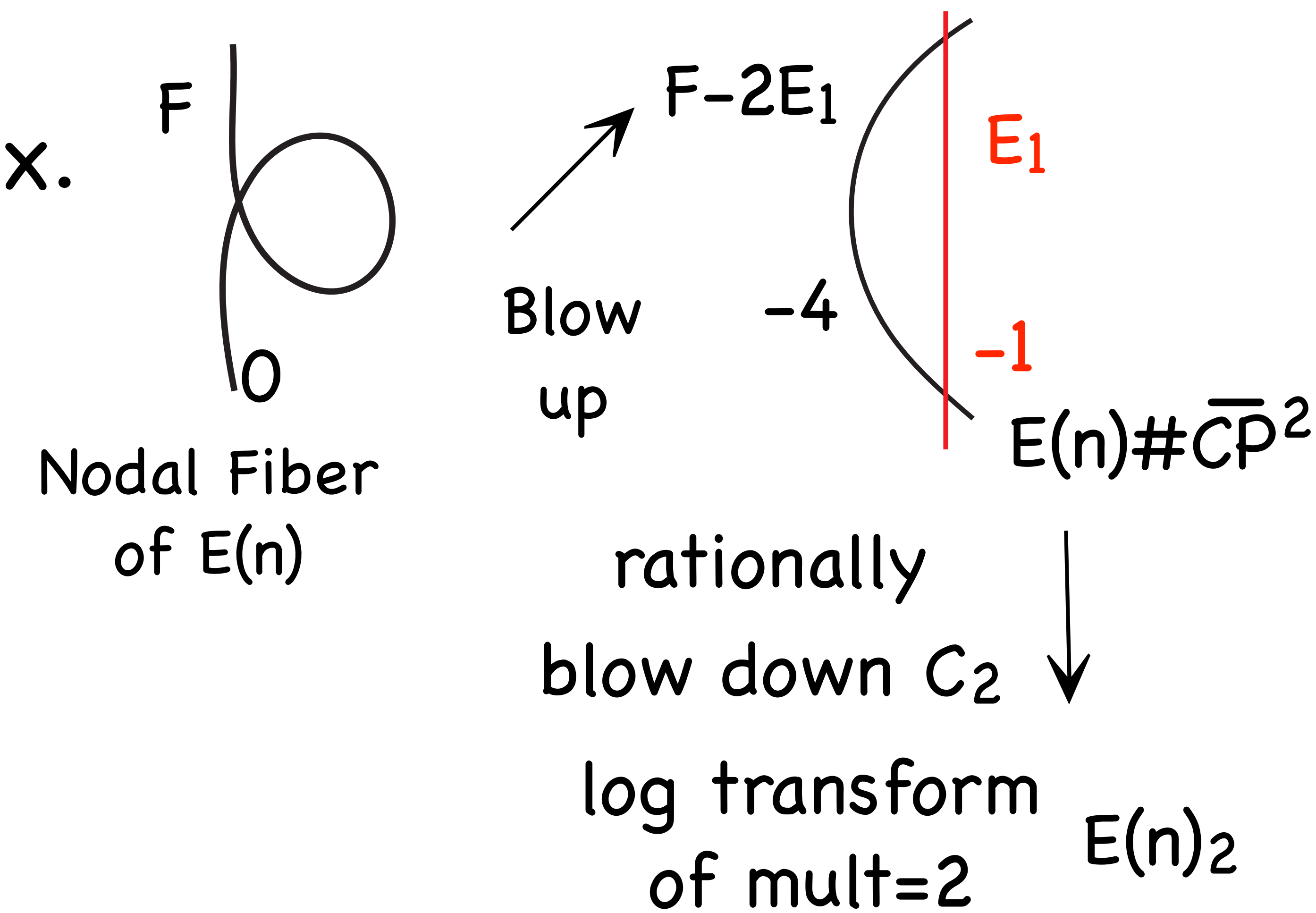
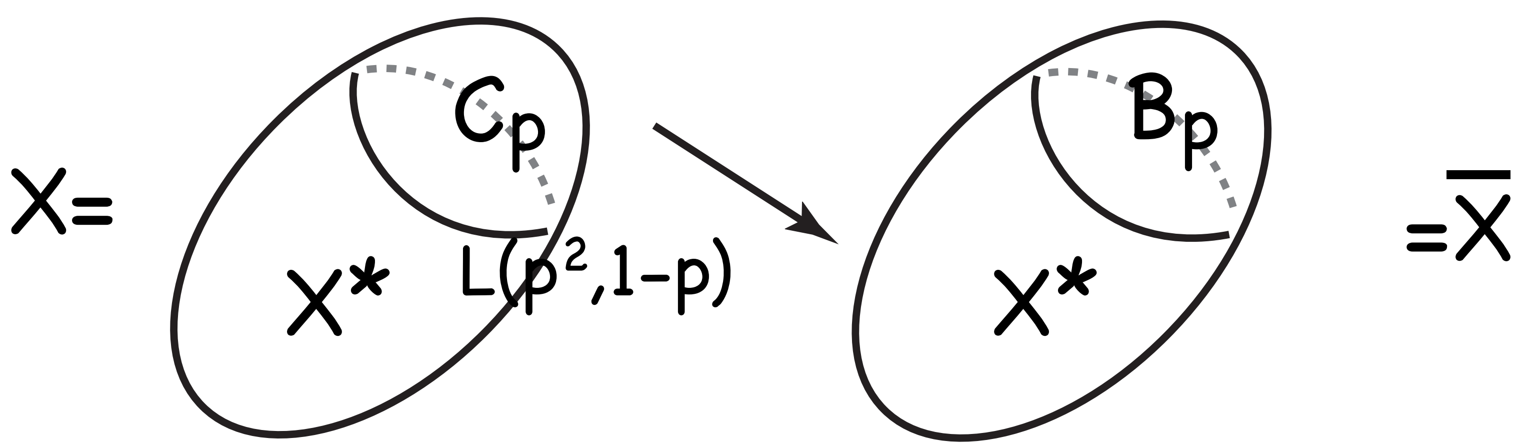


Rational Blowdowns, II



If $\overline{k}$=char homology class in $\overline{X}$, $\exists$ lift k in $X \ni$ PD's agree on X^*

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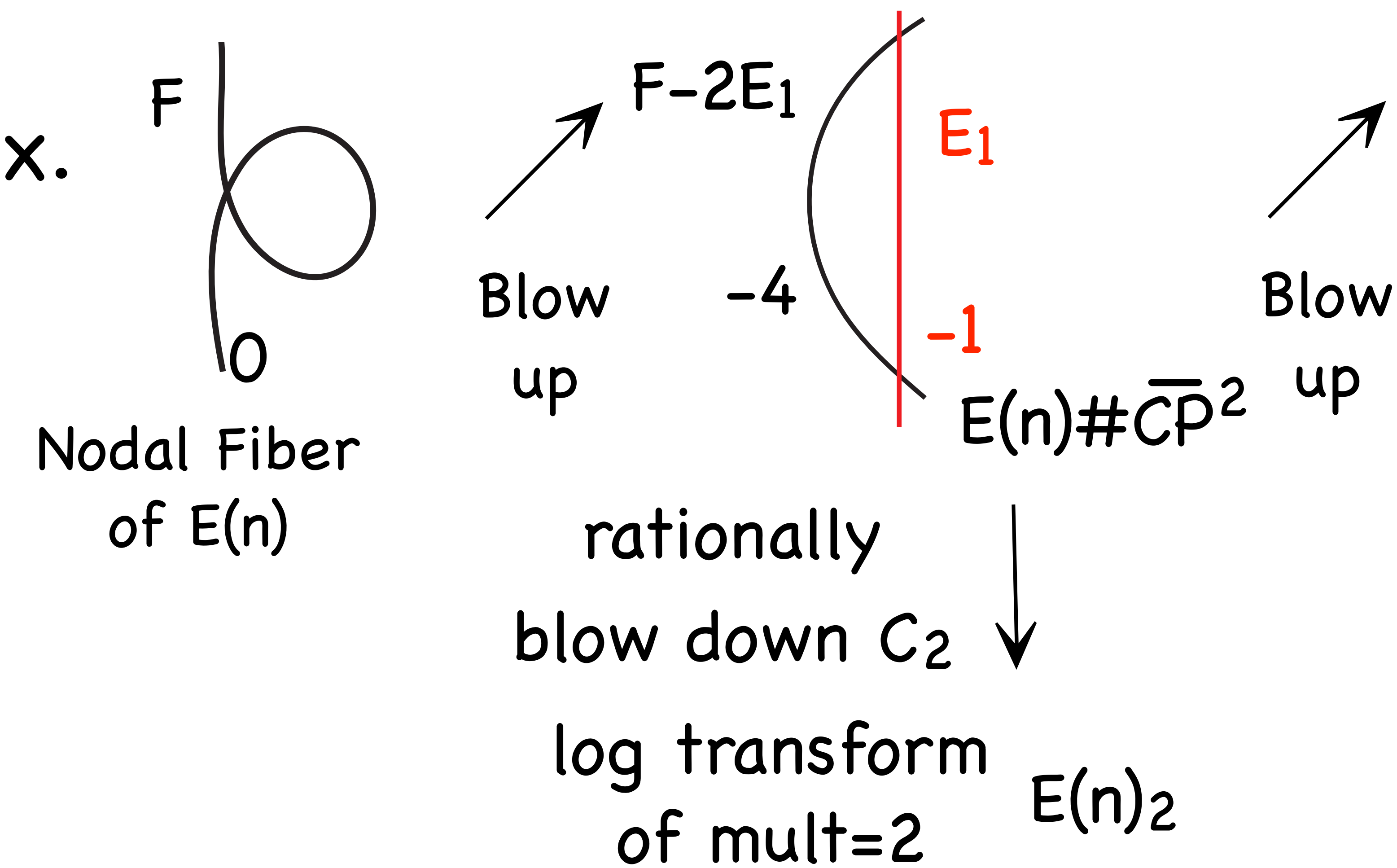

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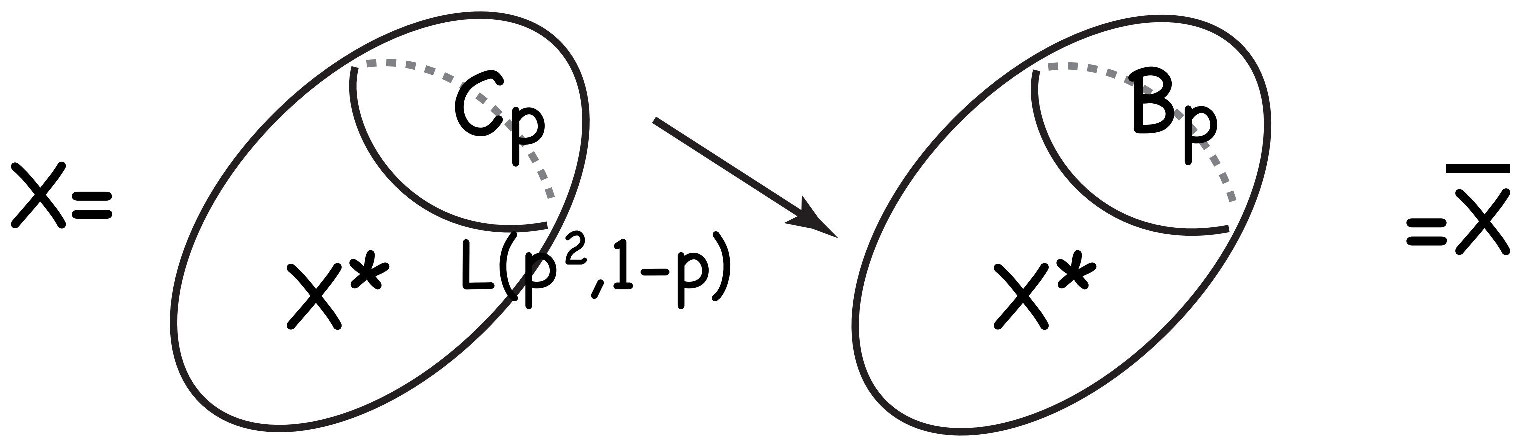

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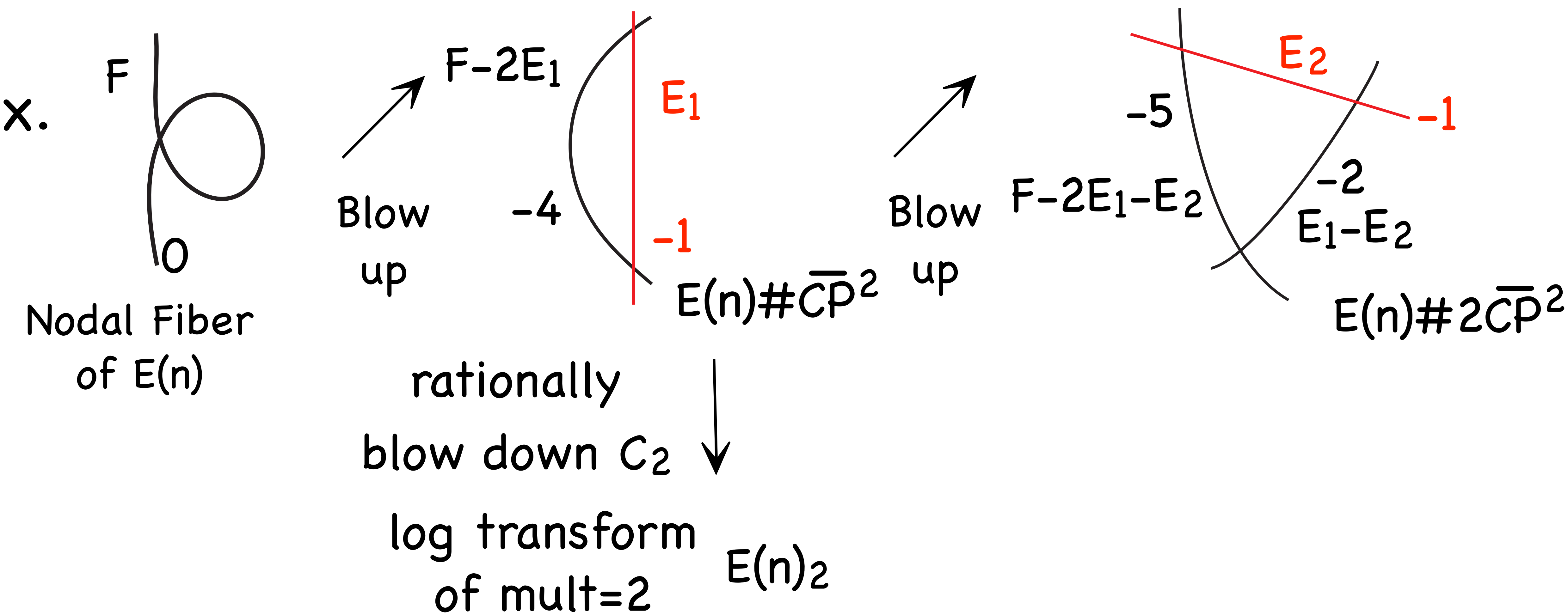

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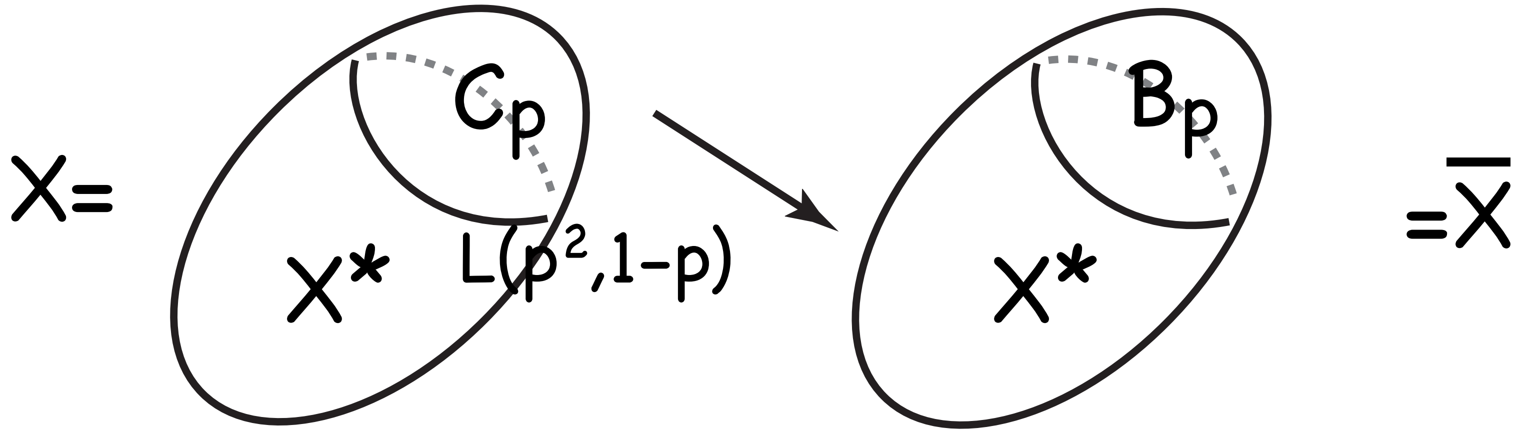

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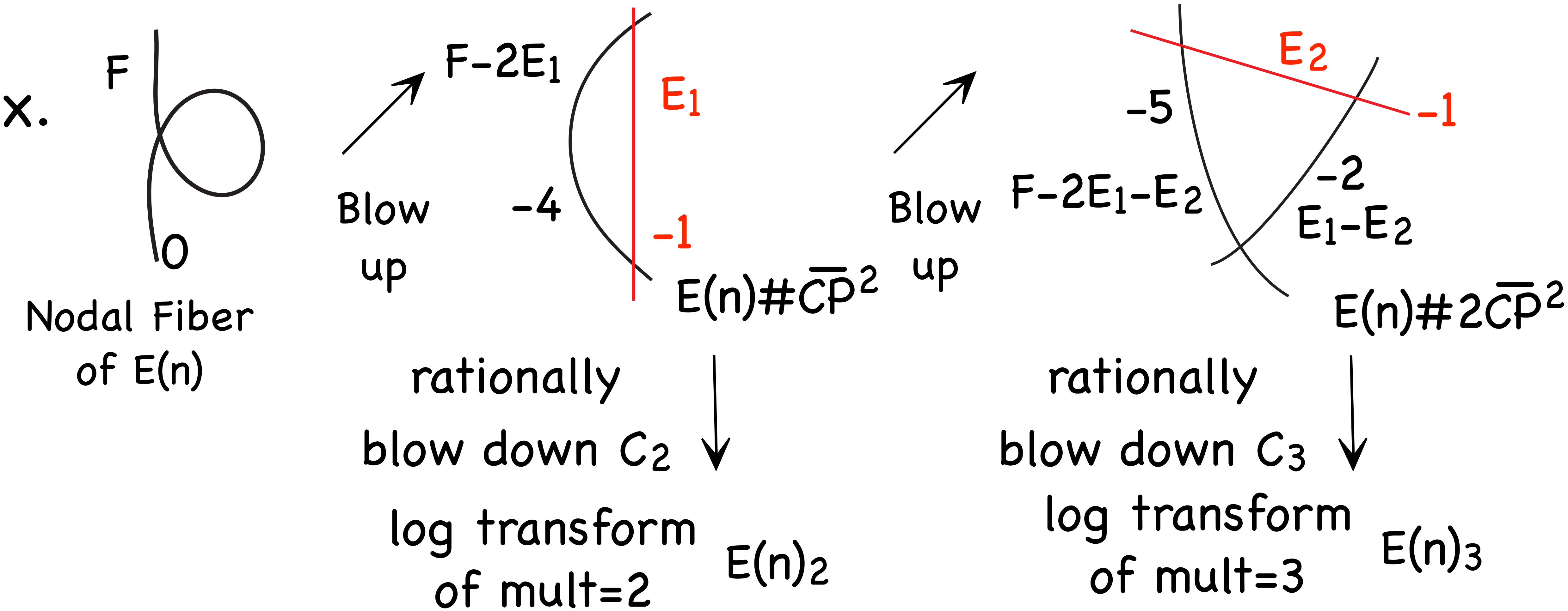
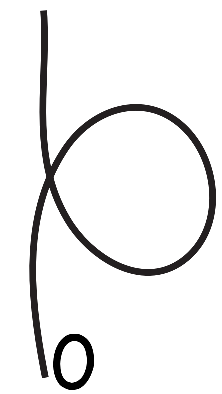


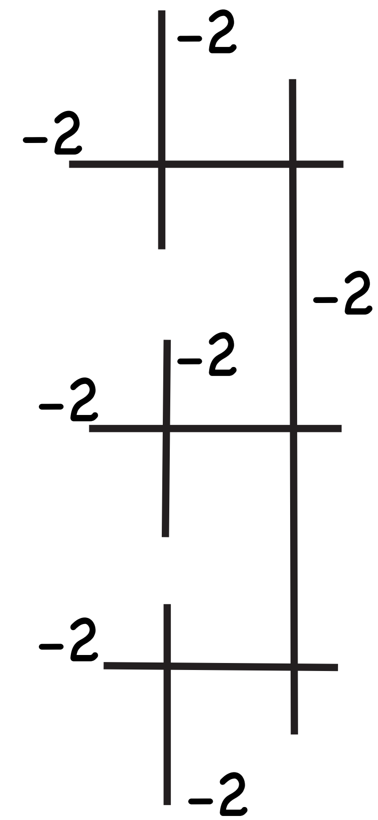
Jongil Park's Idea (2004)

Jongil Park's Idea (2004)

Some singular elliptic fibers



Nodal Fiber



$\tilde{E}_6$ fiber

Jongil Park's Idea (2004)

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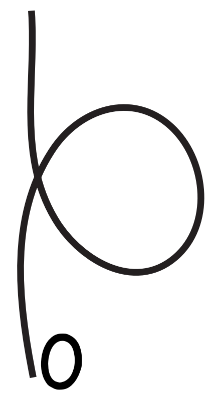

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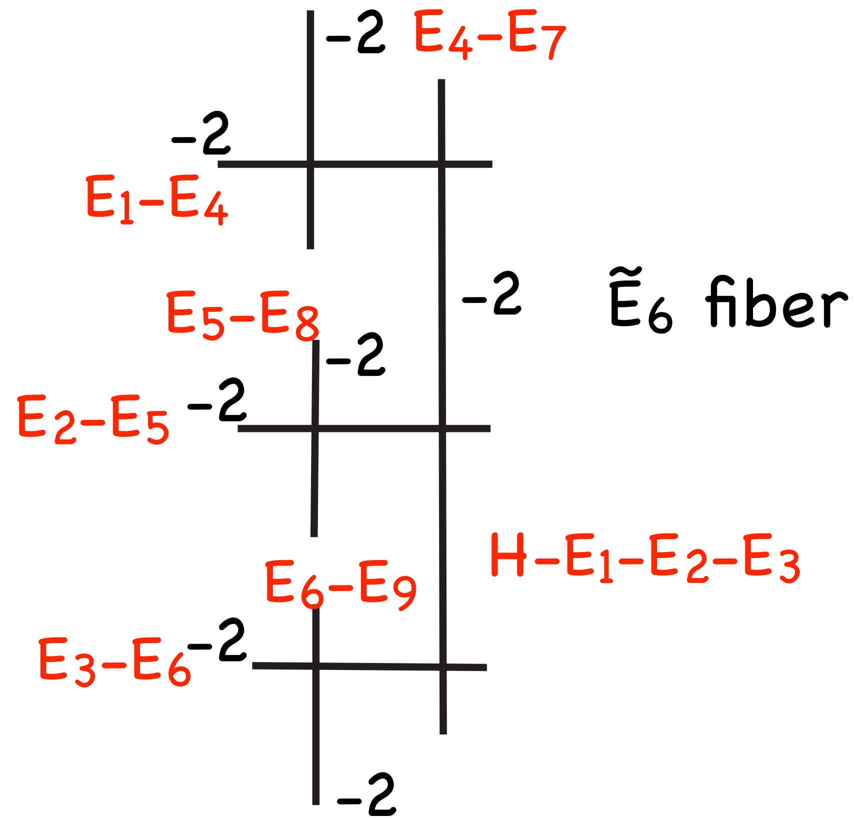

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

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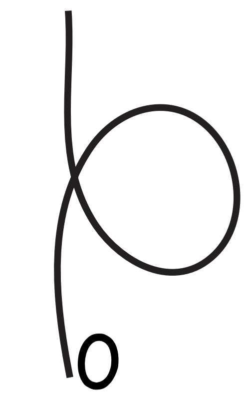

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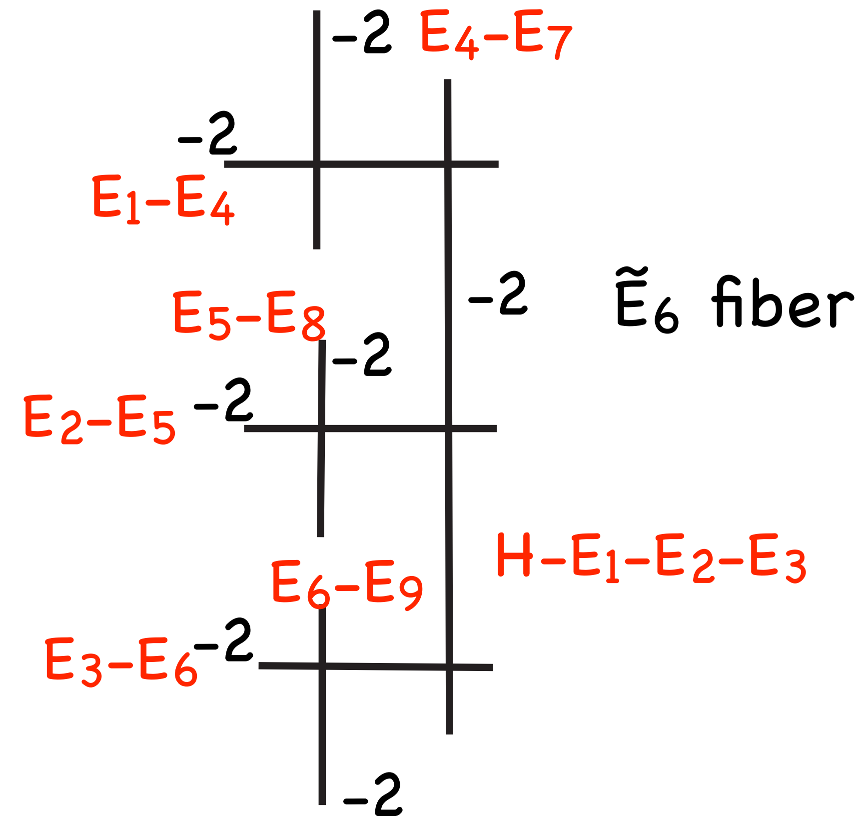

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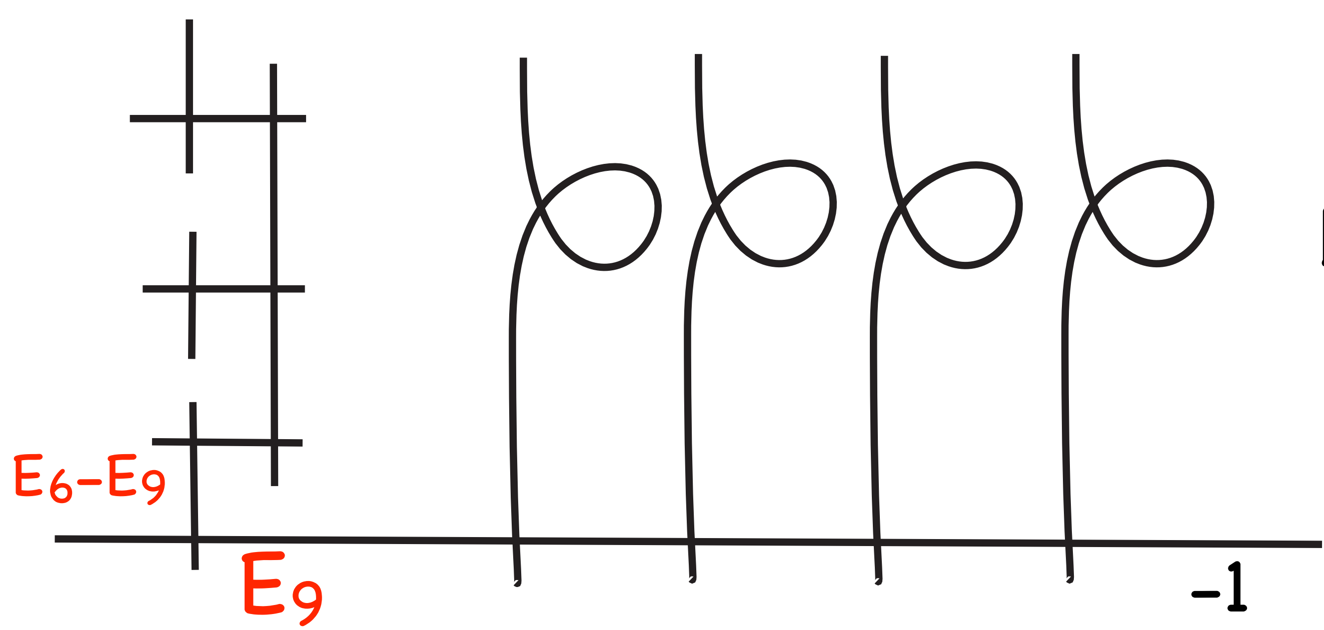

$b^+=1,\ b^-=9$

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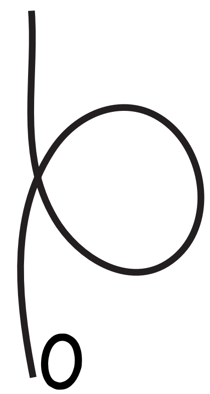

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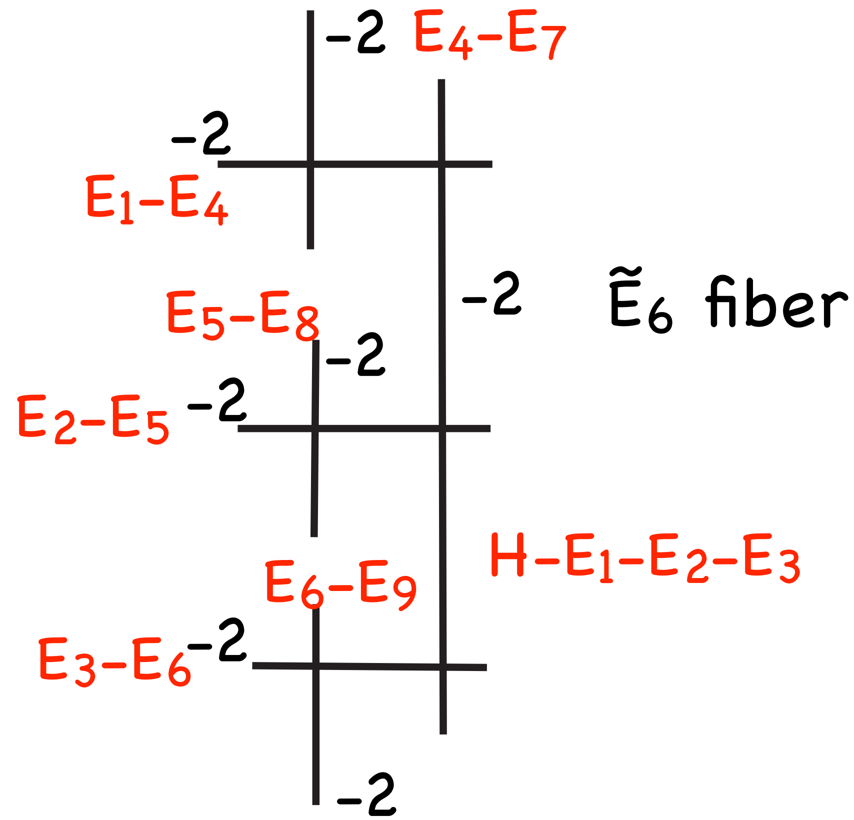

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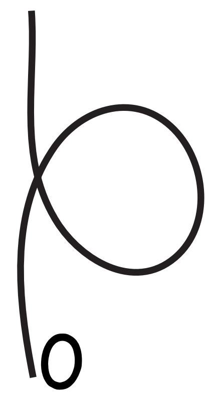

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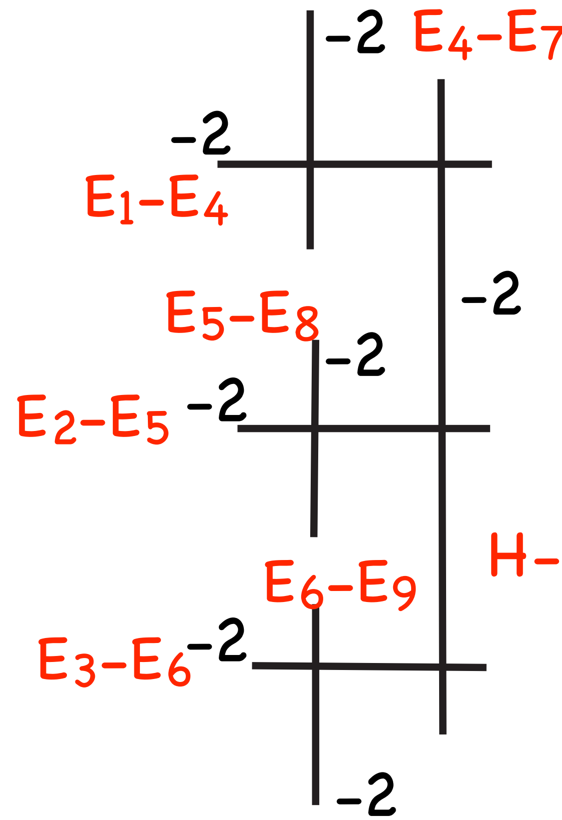


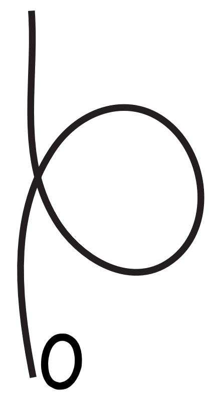
$\tilde{E}_6$ fiber

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

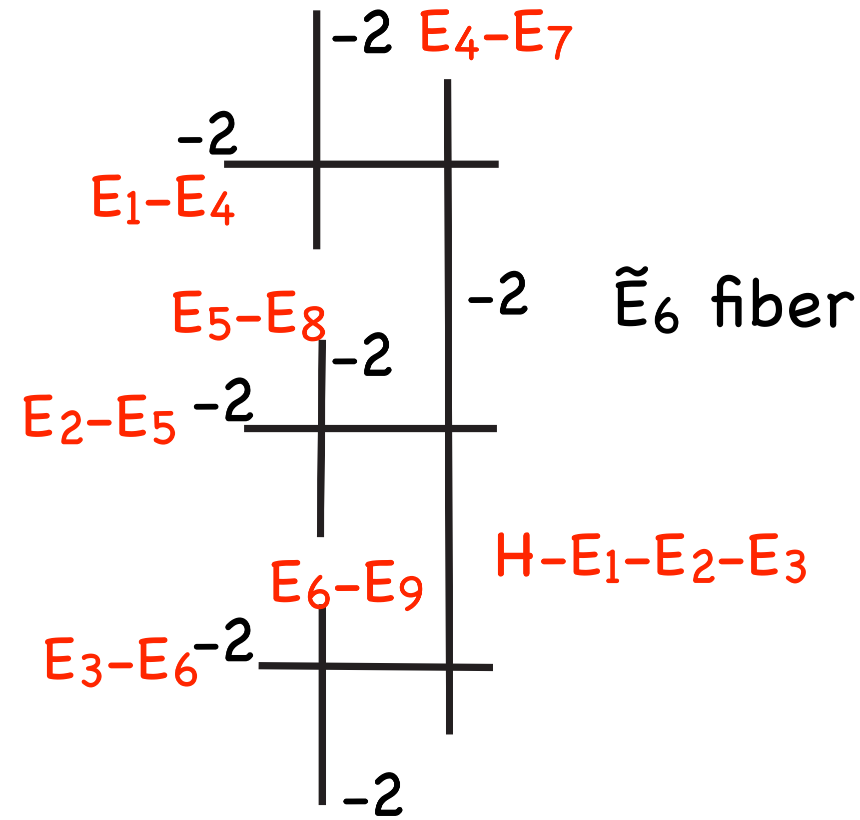
Blow up 4 times

Jongil Park's Idea (2004)

Some singular elliptic fibers



Nodal Fiber



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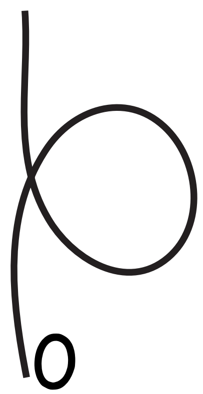

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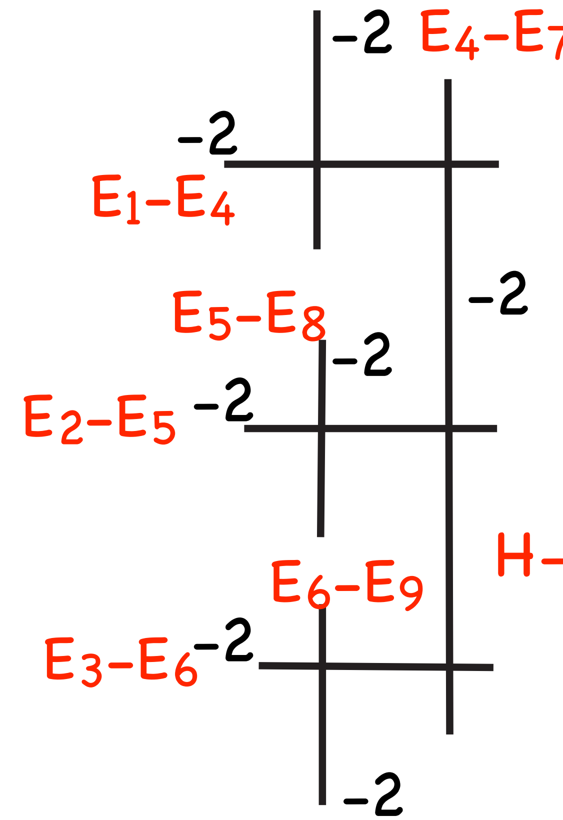
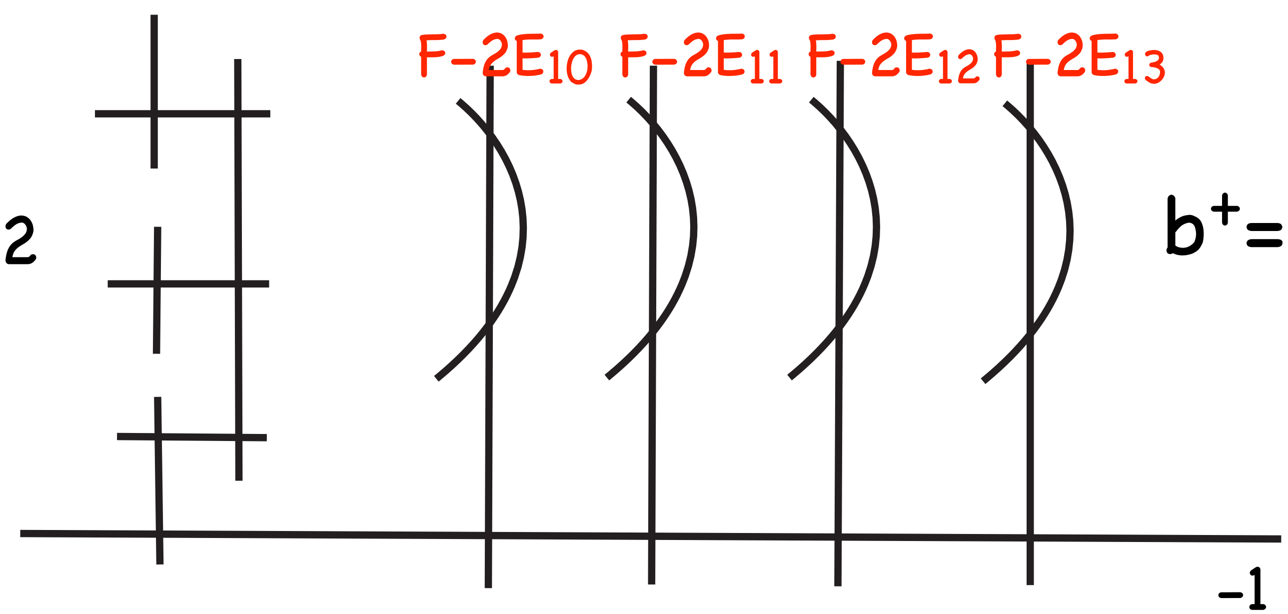


$\tilde{E}_6$ fiber

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

$E(1)\#4\overline{CP}^2$



$b^+=1$, $b^-=13$

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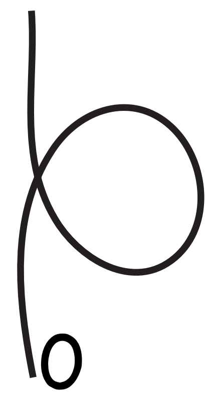

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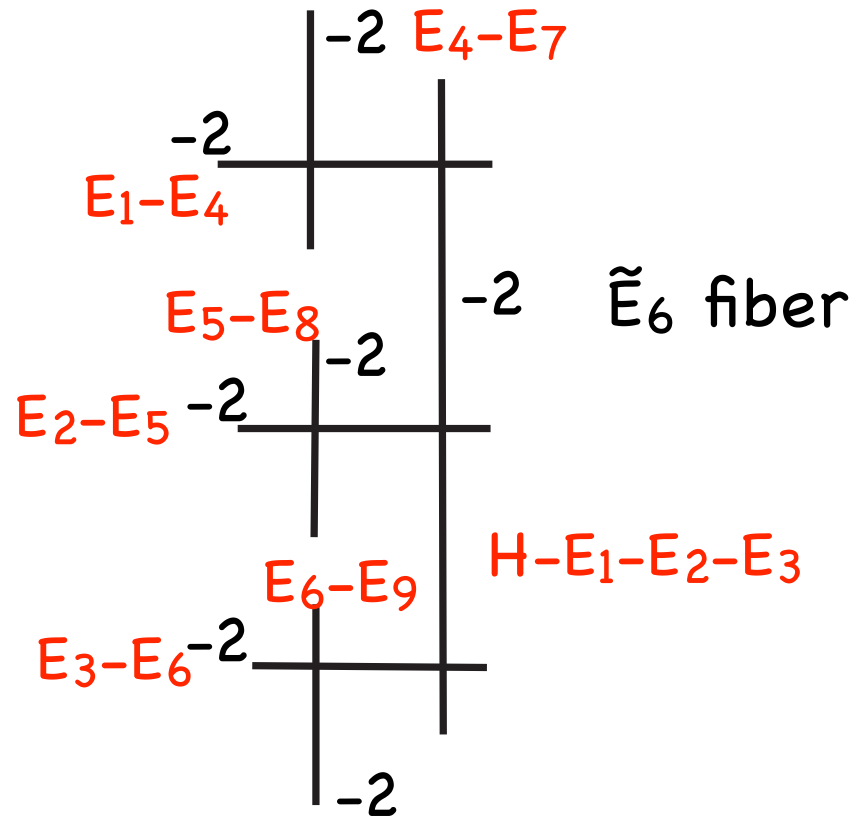

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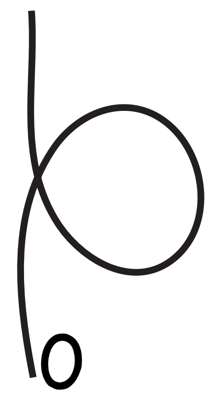

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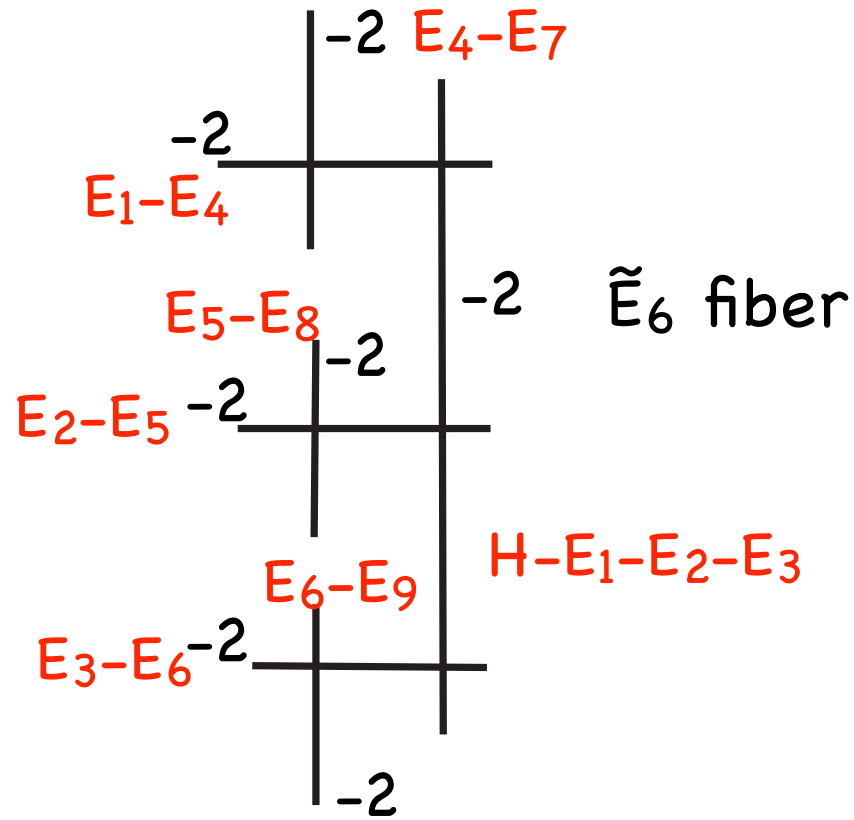
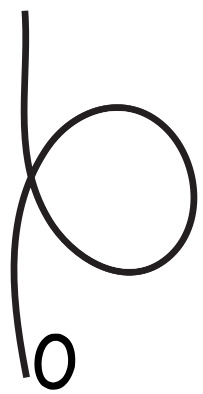


E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

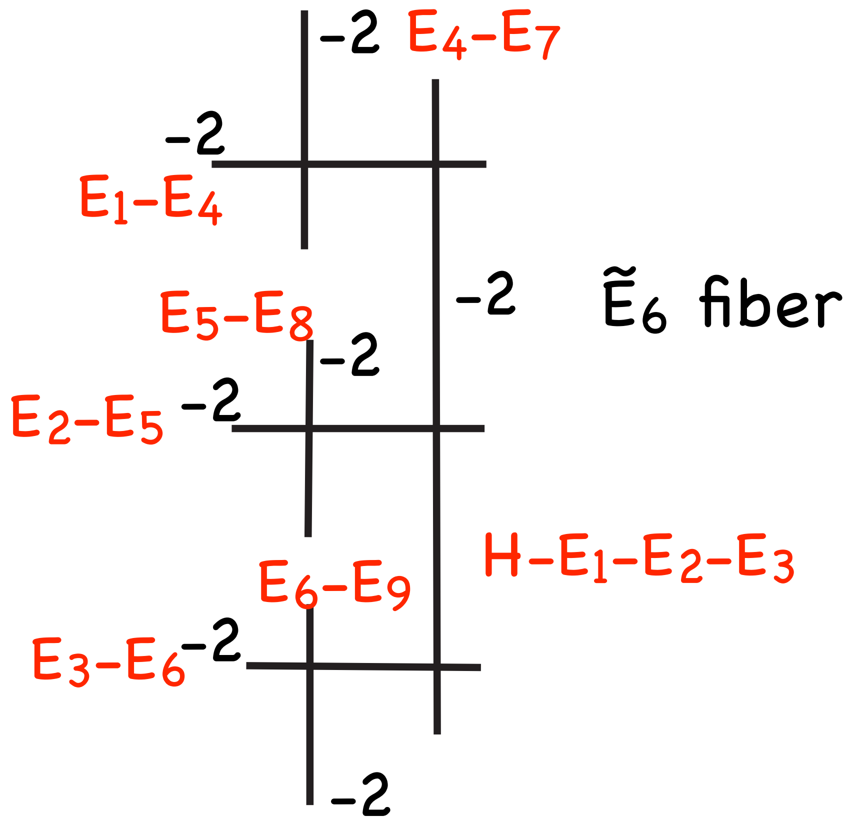
Resolve double points

Jongil Park's Idea (2004)

Some singular elliptic fibers



Nodal Fiber



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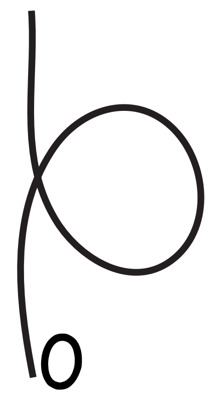

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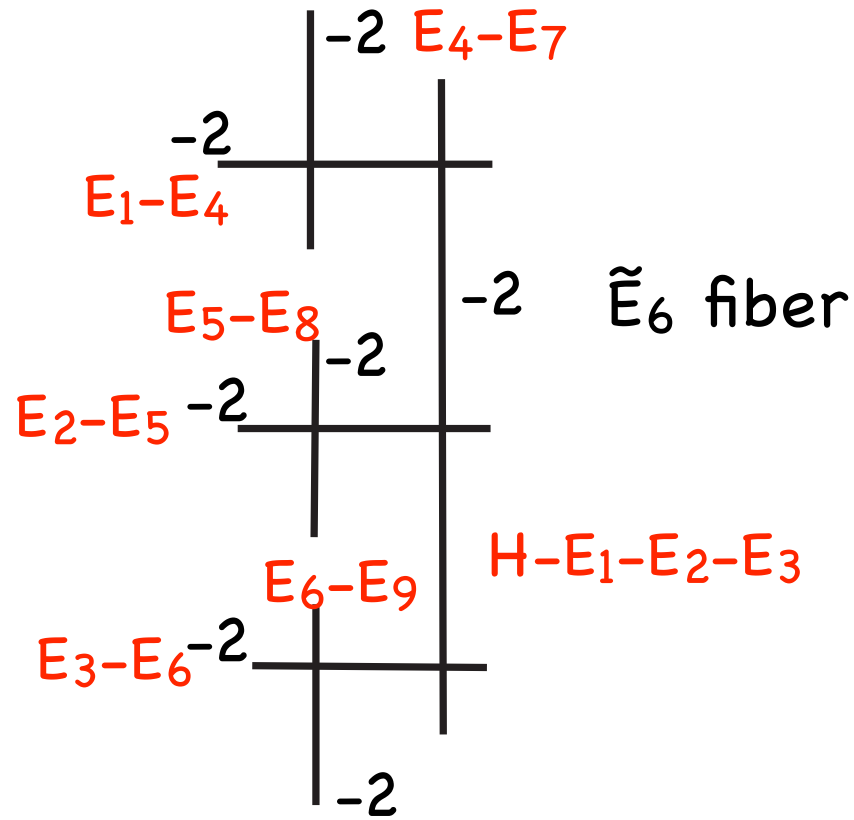
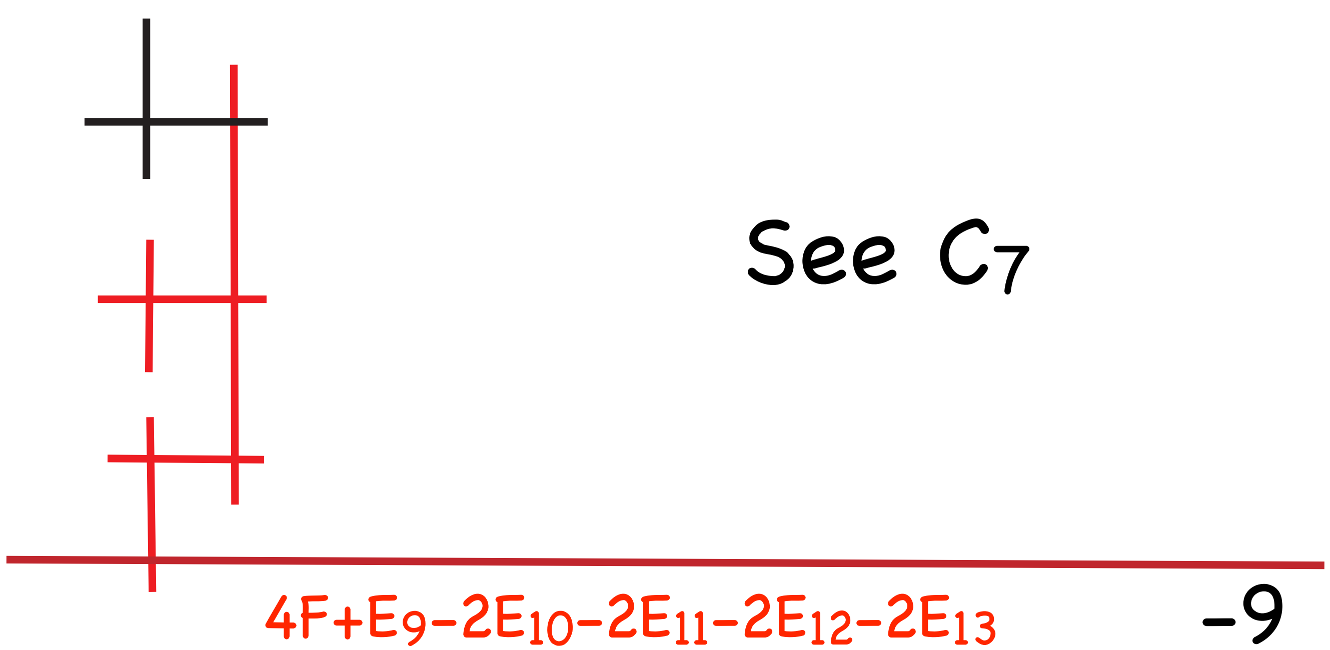


E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

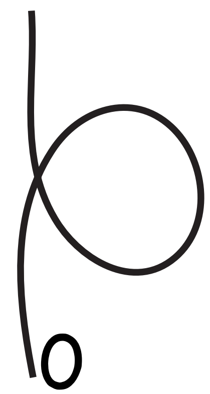
$E(1)\#4CP^2$



See C_7

Jongil Park's Idea (2004)

Some singular elliptic fibers



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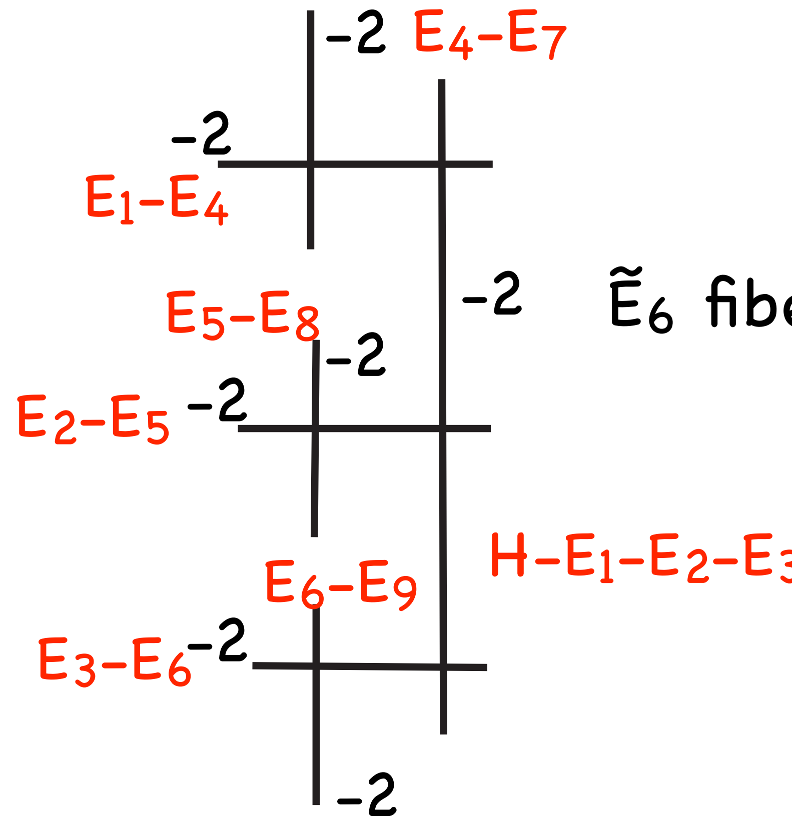

$\tilde{E}_6$ fiber

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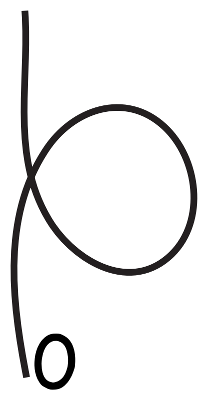

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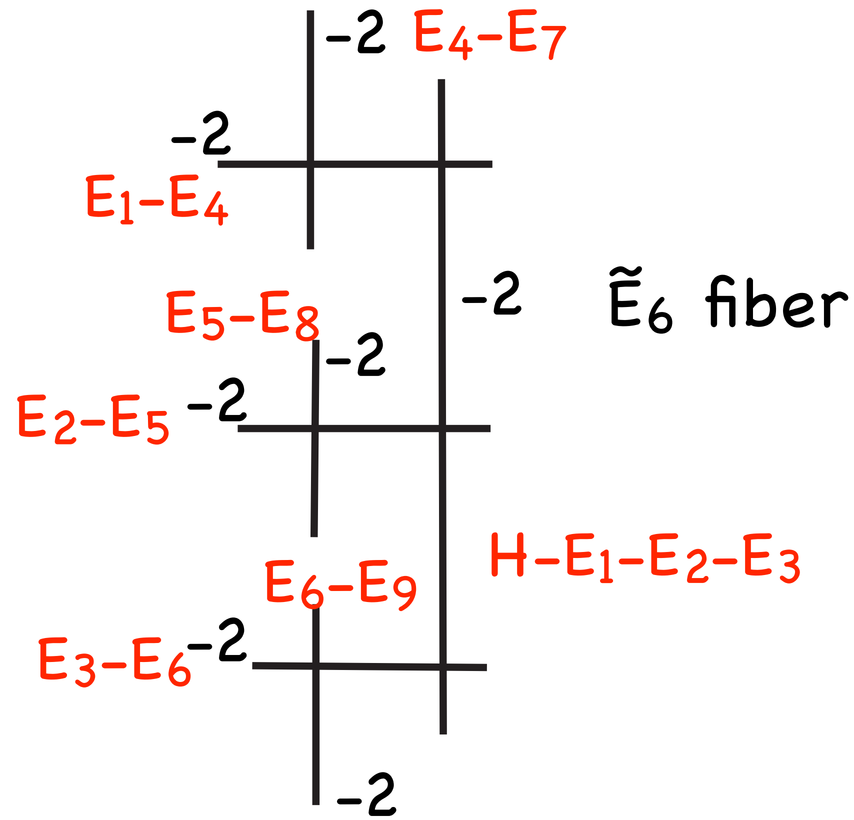

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

Rationally blow down to get Park mfd P

$b^+=1$, $b^-=13-6=7$ and simply connected &

SW≠0 - an exotic $CP^2\#7\overline{CP}^2$

Simply Conn. 4-Mfds w/ $b^+=1$ & $b^-=5,6,7$

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- P homeo to $CP^2\#7\overline{CP}^2$ (& symplectic), $SW_P=t-t^{-1}$

  (This $\Rightarrow$ minimality by blowup formula)

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- Cx Surfaces of general type (J. Park & coauthors)

- These techniques do not seem to work for $b^-<5$

Simply Conn. 4-Mfds w/ $b^+=1$ & $b^-\leq 4$

back to surgery on tori, but -

<u>Prop</u>. If $b^+_X=1$, $b^-_X\leq 8$, $SW_X\neq 0$, $\nexists$ essential torus of square 0 in X.

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If we could surger an essential torus (log transf or knot surgery)of square 0 in a mfd with $b^+=1$ and $b^-\leq 8$ to get $SW\neq 0$, we would be in the situation above.

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So we need to work with nullhomologous tori.

The Morgan, Mrowka, Szabo Surgery Formula

The Morgan, Mrowka, Szabo Surgery Formula

$T \subset X$: torus of square 0

N_T=nbd, $\partial N_T = T^3$ basis $\alpha, \beta, \gamma = \partial D^2$ for $H_1(\partial N_T)$

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$X_{\{p,q,r\}} = (X - N_T) \cup_{\varphi_{pqr}} (T^2 x D^2)$

where $(\varphi_{pqr})_* [\partial D^2] = p\alpha + q\beta + r\gamma$

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Write $SW_{X_{\{p,q,r\}}}(k_{pqr})$

for the coefficient of k_{pqr} in $SW_{X_{\{p,q,r\}}}$

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M-M-Sz Formula:

$\Sigma SW_{X_{\{p,q,r\}}}(k_{pqr})$ =

$$p \Sigma SW_{X_{\{1,0,0\}}}(k_{100}) + q \Sigma SW_{X_{\{0,1,0\}}}(k_{010}) + r \Sigma SW_{X_{\{0,0,1\}}}(k_{001})$$

The Morgan, Mrowka, Szabo Surgery Formula

$T \subset X$: torus of square 0

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The Morgan, Mrowka, Szabo Surgery Formula

$T \subset X$: torus of square 0

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"$SW_{X_{\{p,q,r\}}} = pSW_{X_{\{1,0,0\}}} + qSW_{X_{\{0,1,0\}}} + rSW_{X_{\{0,0,1\}}}$"

- $X_{\{0,0,1\}} = X$, and $X_{\{1,0,0\}}$ and $X_{\{0,1,0\}}$ are results of $S^1 \times$ 0-surgeries

Using the M-M-Sz Formula

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Ex: Want to construct exotic smooth str's on $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$

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Need to find useful nullhomologous torus T

in mfd with $b^+=1$, $b^-=n$, $\pi_1=0$

Using the M-M-Sz Formula

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(0,k,1)-surgery = S^1x(1/k-Dehn surgery)

has same homology as X if $\beta=0$ in $H_1(X-T)$, and

$$SW_{X_{\{0,k,1\}}} \sim k\,SW_{X_{\{0,1,0\}}} + SW_{X_{\{0,0,1\}}=X}$$

0-surgery

Using the M-M-Sz Formula

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Using the M-M-Sz Formula

Ex: Want to construct exotic smooth str's on $CP^2\#n\overline{CP}^2$

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How to achieve this?

Surgery on Tori

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$T' \subset X'$ $\alpha', \beta', \gamma' = \partial D^2$

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This is how we can assure $SW_{X'} \neq 0$.

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Get ∞ family if all $1/k$-surgeries are s.c.

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Baldridge-Kirk
Akhmedov-Park

Their models constructed
by cut-and-paste

How to find model manifolds

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(Stern and I have shown how to do this in $CP^2\#n\overline{CP}^2$ for $2\leqq n\leqq 7$.)

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Need to look for tori to surger in $CP^2 \# n\overline{CP}^2$

such that genus of K is "forced up"

$CP^2 \# 3\overline{CP}^2$ (again)

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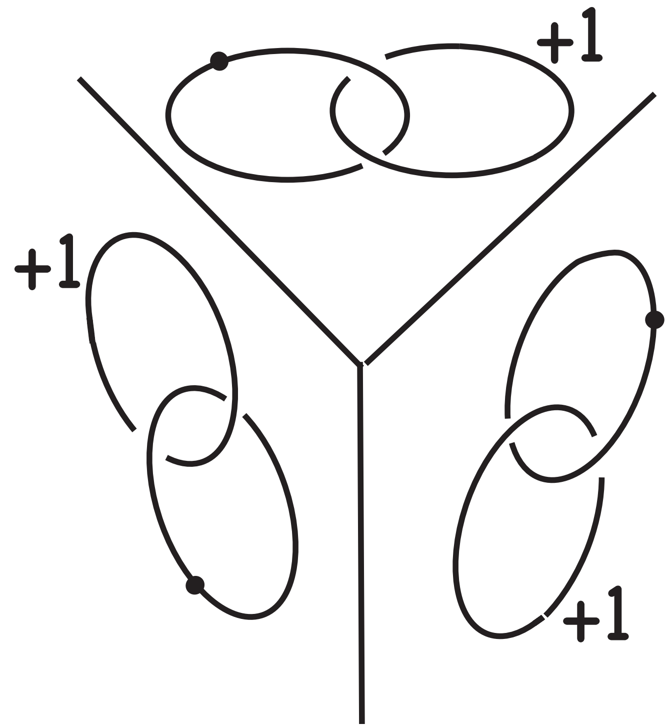

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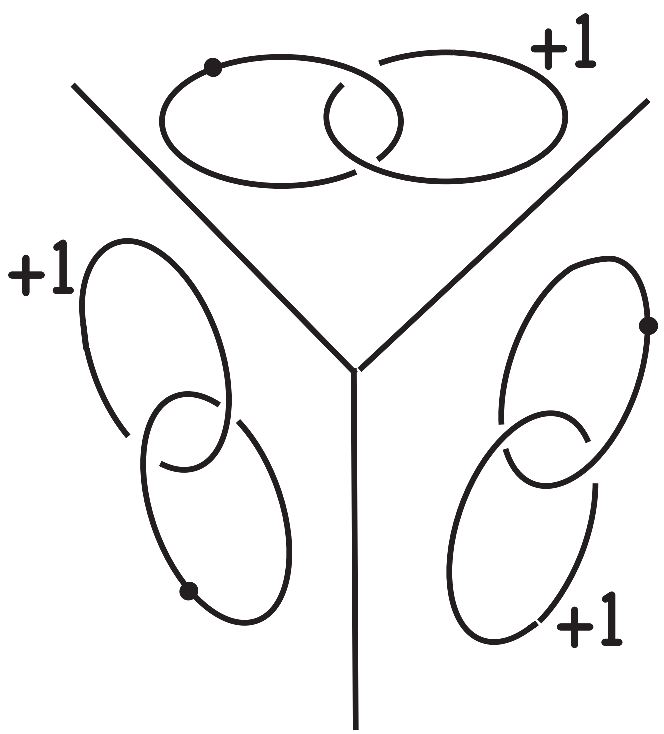

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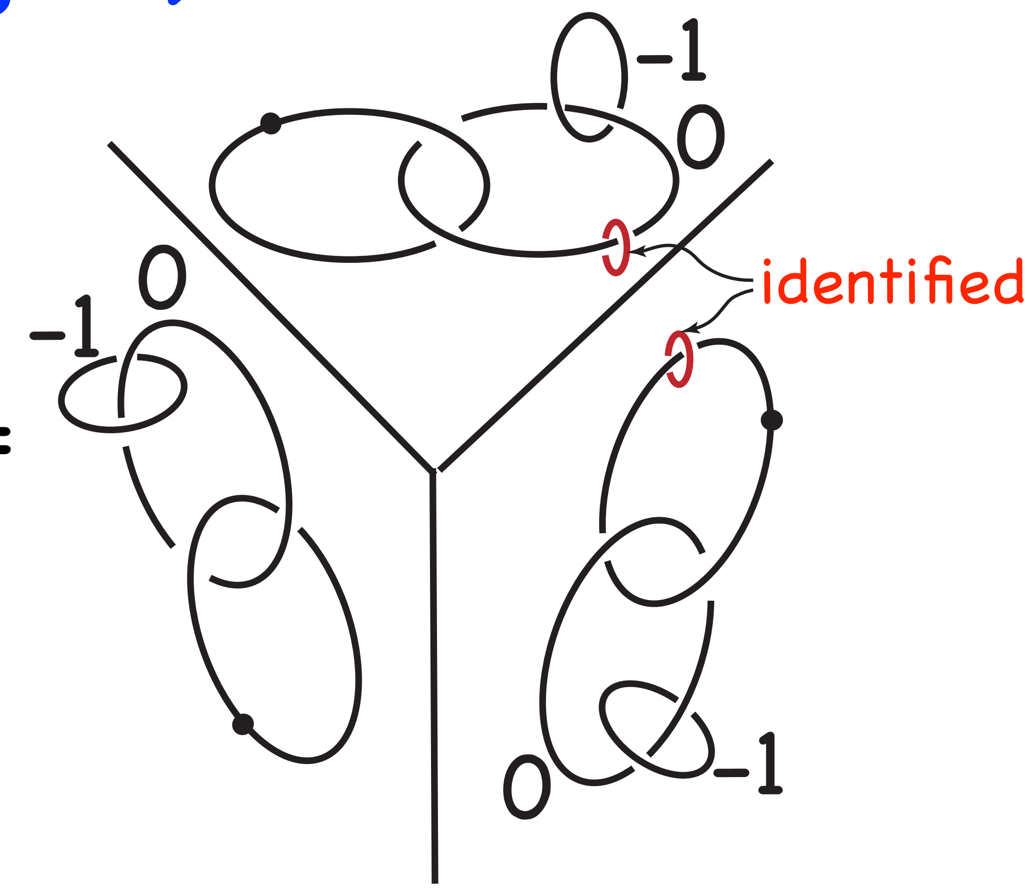

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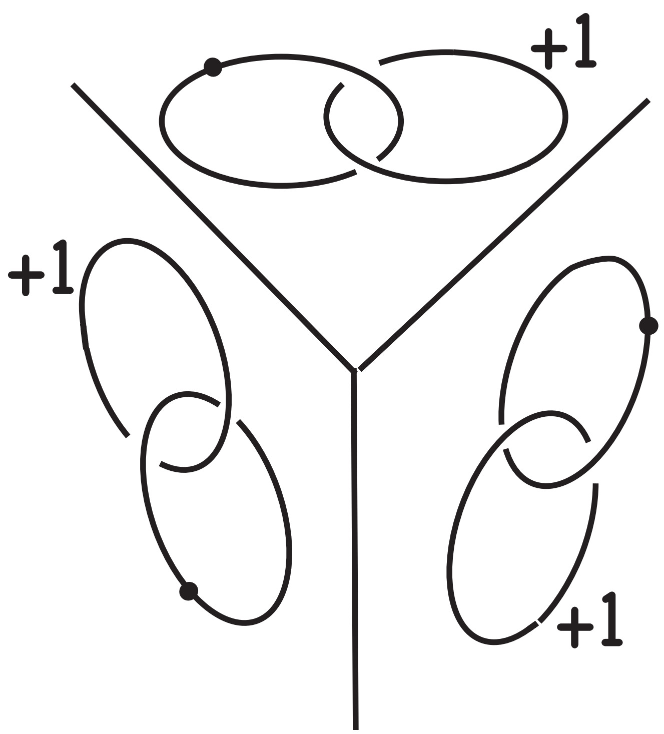

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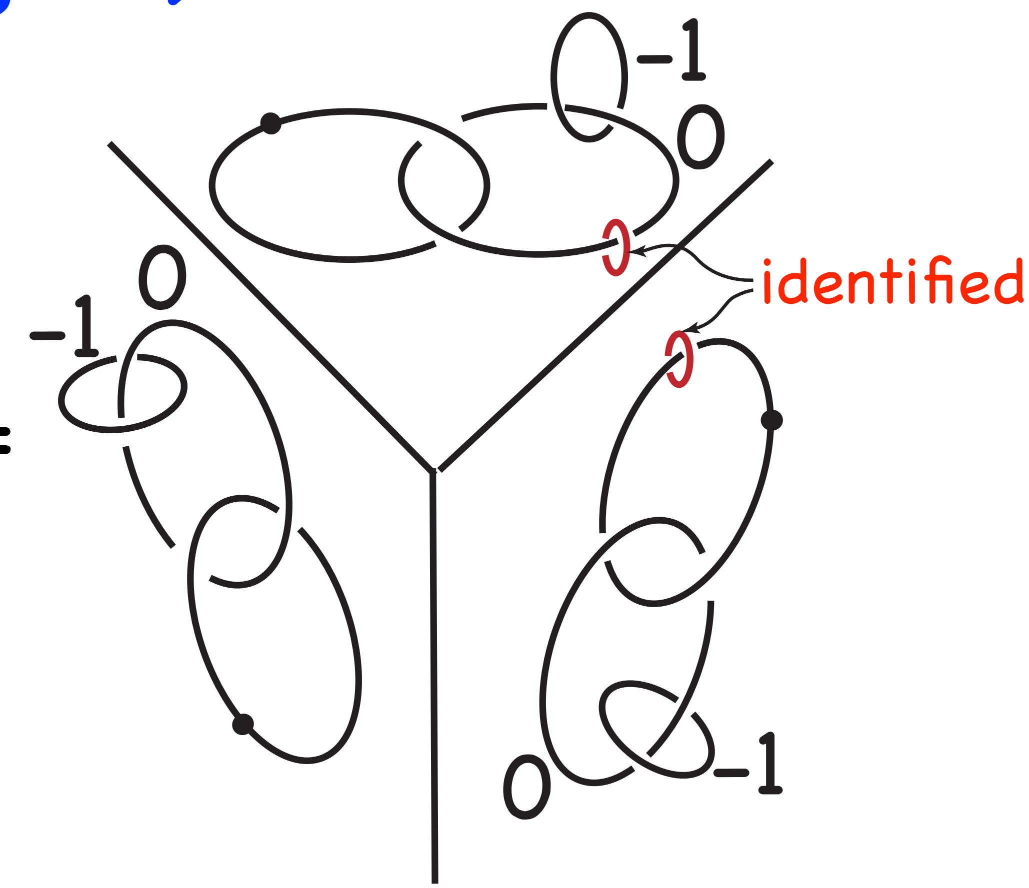

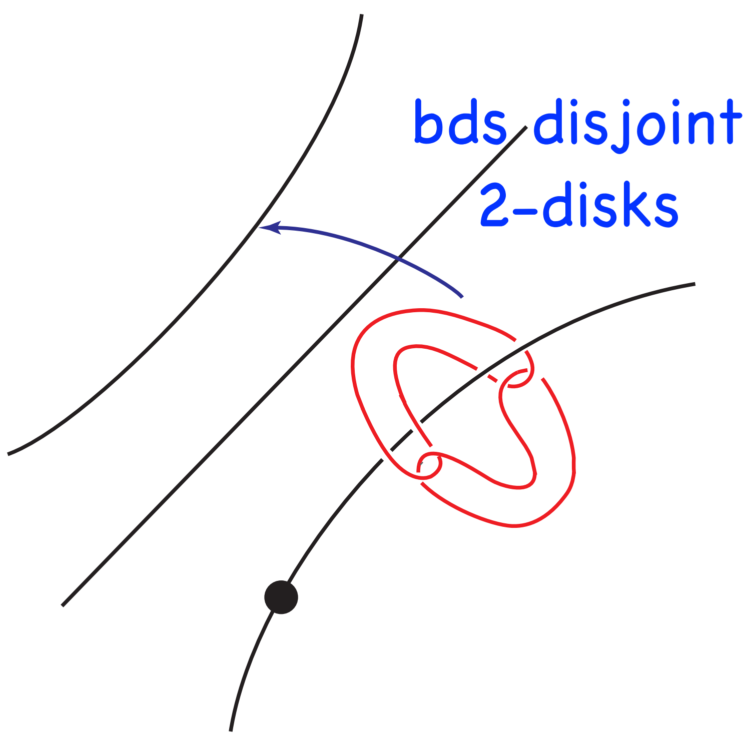

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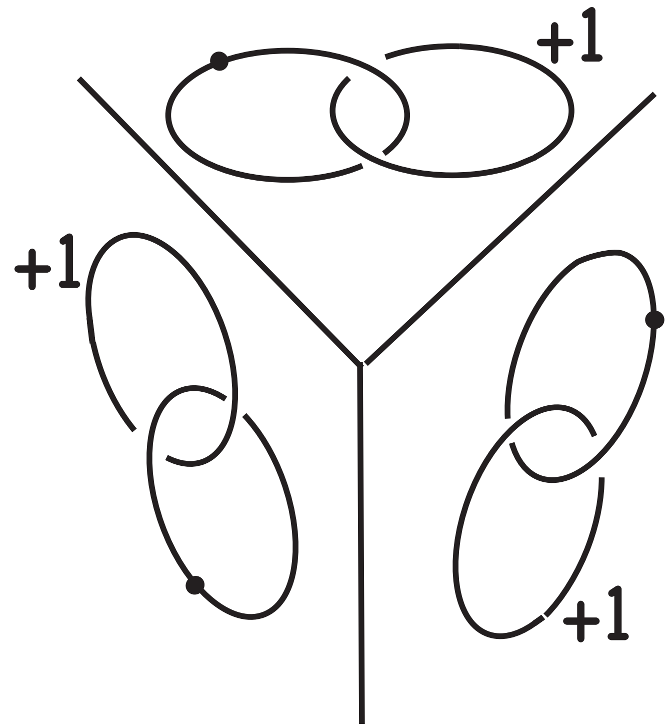

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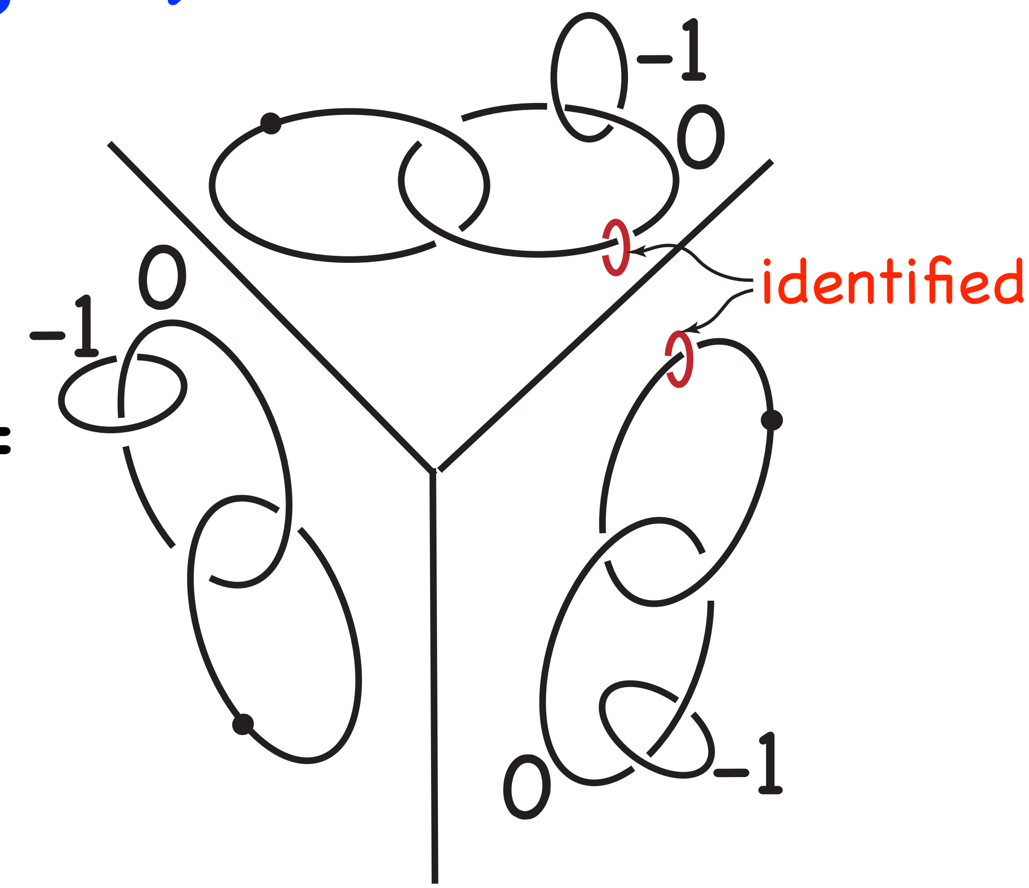

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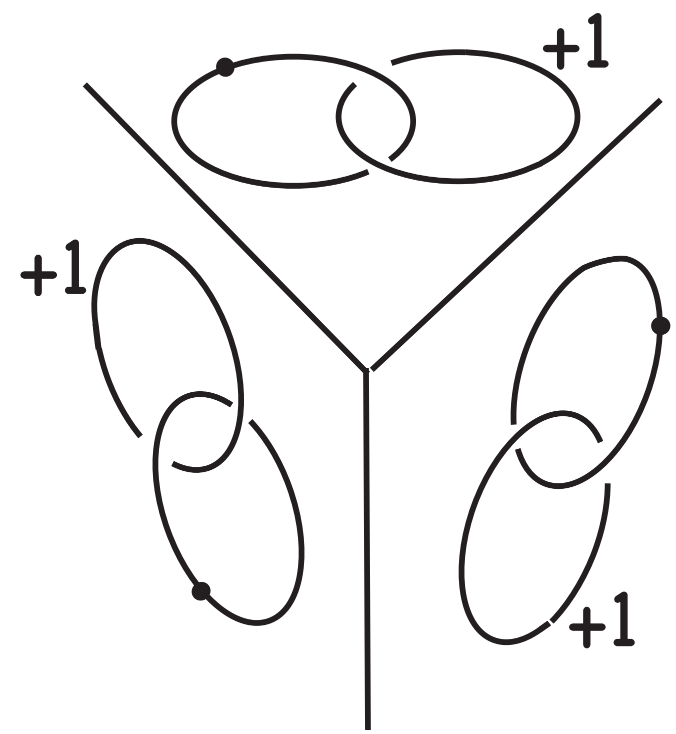

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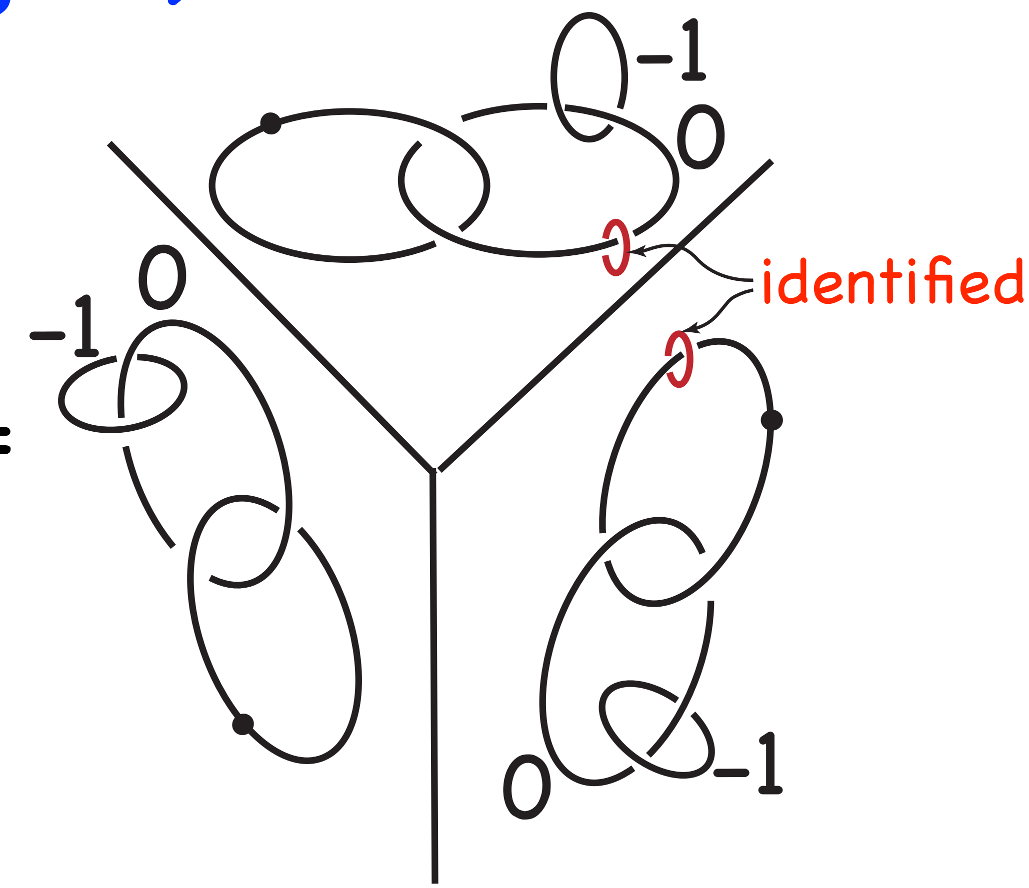
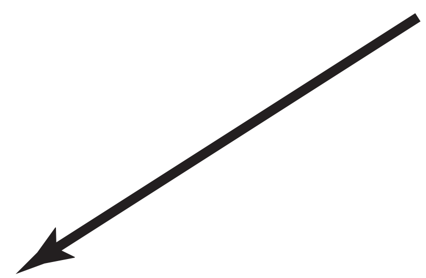
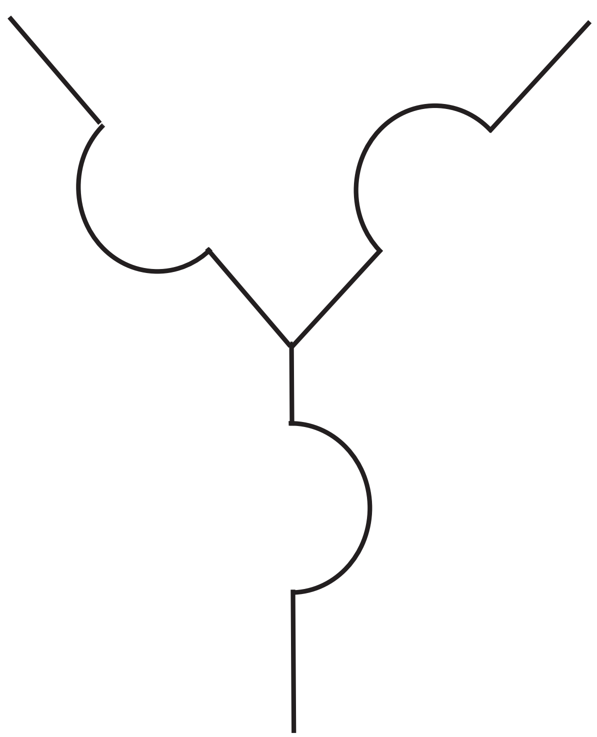
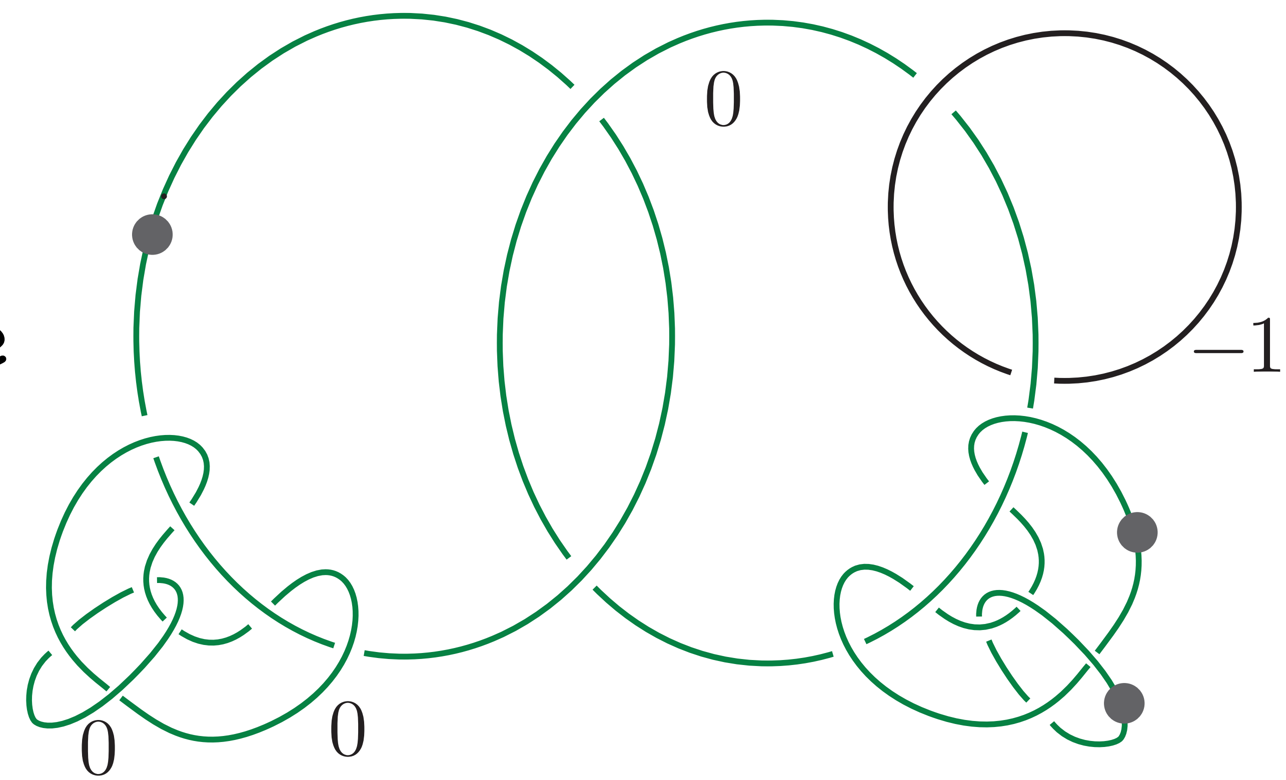






each
piece



Bing Tori

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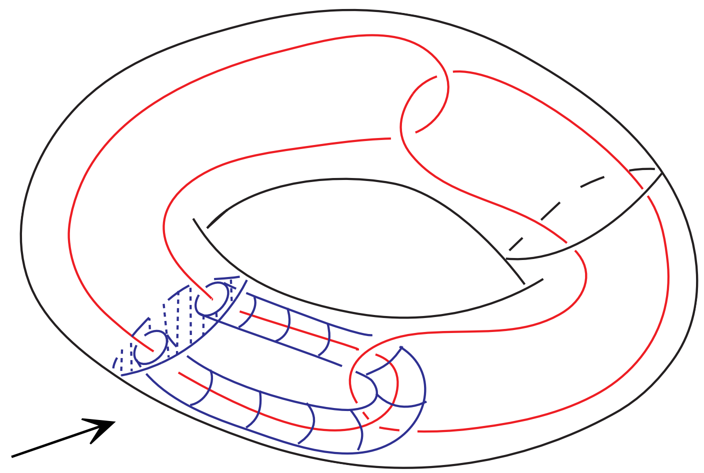
A=Green part= $S^1 \times$



$- pt \times$

$= T^2 \times D^2 - pt \times T_0$

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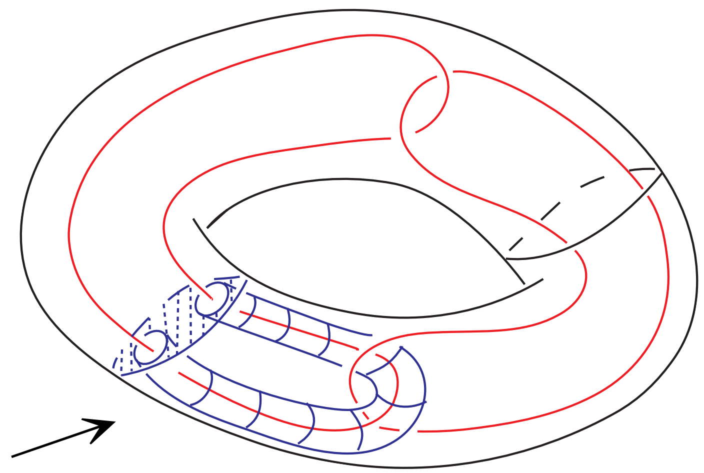
A=Green part= $S^1 \times$  $= T^2 \times D^2 - pt \times T_0$

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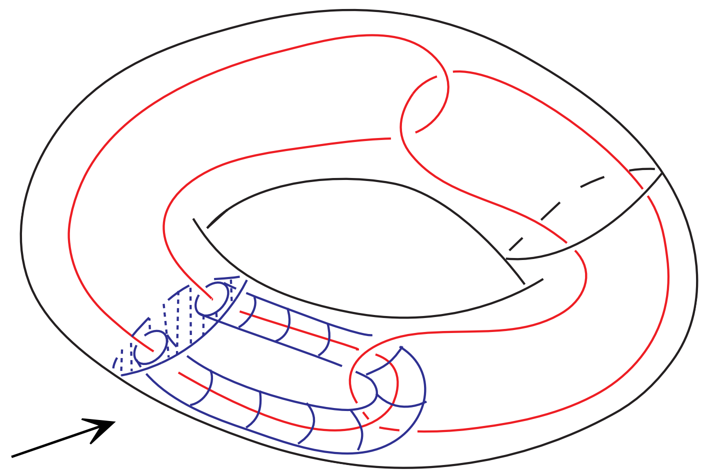

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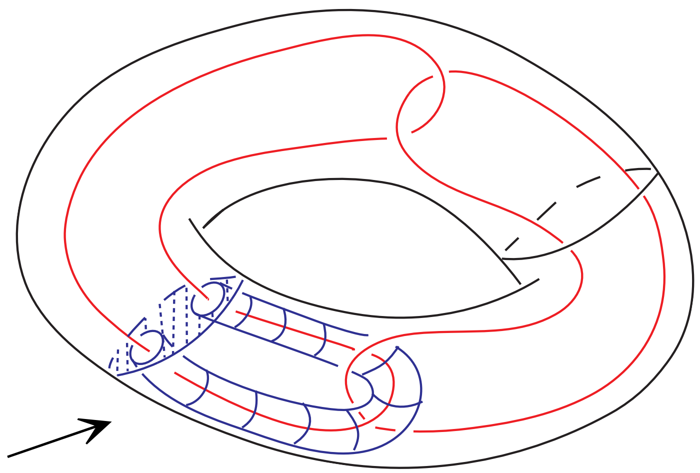

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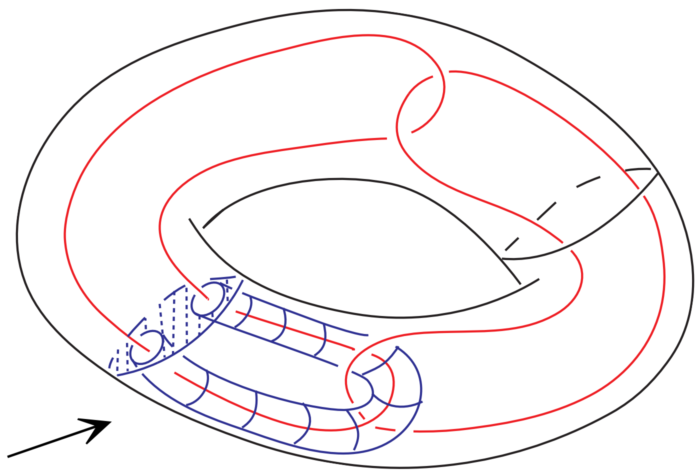

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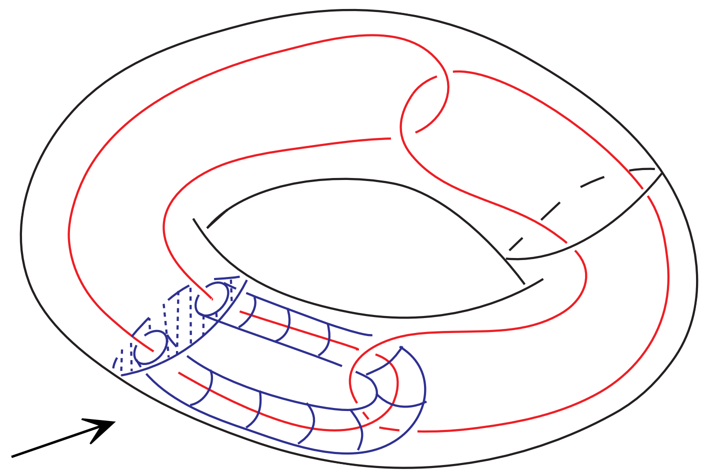
A=Green part= $S^1 \times$  $=T^2 \times D^2 - pt \times T_0$

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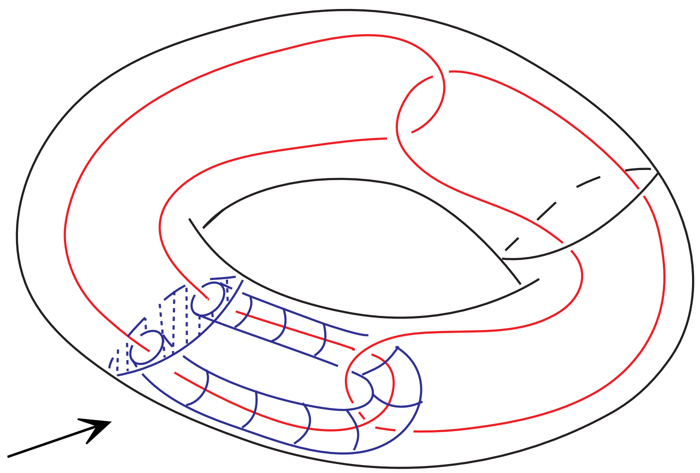
A=Green part= $S^1 \times$  $=T^2 \times D^2 - pt \times T_0$

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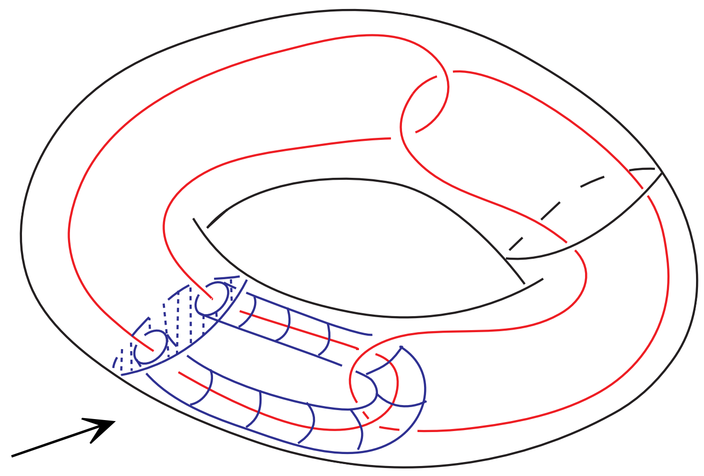
A=Green part= $S^1 \times$  $=T^2 \times D^2$ - pt$\times T_0$

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Moral: Look for useful emb's of A.

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- $\exists$ proposed examples for $S^2 x S^2$
  but π_1 calculation incorrect

These use reverse eng. with model $M = \Sigma_2$ bundle over Σ_2
$\Rightarrow$ M aspherical

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- $\exists$ proposed examples for $S^2 \times S^2$
  but π_1 calculation incorrect

These use reverse eng. with model $M=\Sigma_2$ bundle over Σ_2
$\Rightarrow$ M aspherical

<u>Conjecture</u>: The result of Lagrangian (i.e. Luttinger) surgery on a symplectically aspherical 4-mfd is again sympl. asph. ($\Rightarrow$ π_1 infinite).

Current State of Affairs

- $\exists$ exotic smooth str's on $CP^2\#n\overline{CP}^2$, $2\leqq n\leqq 9$
- $\exists$ exotic smooth str's on $CP^2\#n\overline{CP}^2$, $n\geqq 10$
  (but no examples minimal, and $c_1^2<0$ for these)
- Open: CP^2, $CP^2\#\overline{CP}^2$, S^2xS^2
- $\exists$ proposed examples for S^2xS^2
  but π_1 calculation incorrect

These use reverse eng. with model $M=\Sigma_2$ bundle over Σ_2
$\Rightarrow$ M aspherical

Conjecture: The result of Lagrangian (i.e. Luttinger) surgery on a symplectically aspherical 4-mfd is again sympl. asph. ($\Rightarrow$ π_1 infinite).

(Very) Optimistic Conj. Every s.c smooth 4-mfd can be obtained from surgery on tori in a conn. sum of copies of S^4, CP^2, $\overline{CP}^2$, and S^2xS^2.