Math 481

Summer 2012

Test 2

You have one hour and fifty minutes to complete this test. You may not use a computer, phone, or other electronic device to aid you. Be sure to justify all your answers to receive full credit.

1. Let G be a connected planar graph of order less than 12. Prove that $\delta(G) \leq 4$.

Suppose that $\delta(G) \geq 5$. Then if D is the sum of the vertex degrees, we have $D = 2q \geq 5n$. Also, we have that $D = 2q \leq 6n - 12$. For such a graph to be possible, we must have

$$5n \leq 6n - 12 \;\Rightarrow\; n \geq 12.$$

Therefore, if $n < 12$, $\delta(G) \leq 4$.

2. Prove that a graph G is planar if and only if every subdivision of G is planar.

($\Rightarrow$) Let G be planar and let G' be a subdivision of G. Since in each subdivision of an edge of G the path that replaces the edge can be drawn along the original edge, G' is planar also.

($\Leftarrow$) Suppose that G is not planar. Then take any representation of G and form a subdivision of G by selecting an edge and adding a new vertex on that edge at any point except on the intersection with another edge. Then this is a non-planar subdivision.

3. Let T be the set of all labeled trees on n vertices, and let S be the set of all sequences of length $n-2$ with entries from $\{1, \ldots, n\}$. Use Prüfer's algorithms to show that $|T| = |S| = n^{n-2}$.

Prüfer's algorithm part 1 gives a method for assigning a unique sequence from S to each element of T. Therefore, $|T| \leq |S|$. Similarly, Prüfer's algorithm part 2 gives us a method for assigning a unique labeled tree to each element of S. Therefore, $|S| \leq |T|$. Together, these inequalities show that $|S| = |T|$. To build an element of S, we have n choices for each of the $n-2$ entries, so there is a total of n^{n-2} elements in S.

4. Suppose G is a connected, non-complete graph with $\delta(G) \geq 2$. Show that there exists a vertex $v \in V(G)$ with neighbors v_1 and v_2 such that v_1 and v_2 are not

adjacent.

Suppose that this were not true. Then, for every vertex $u \in V(G)$, $\langle N(u) \rangle$ is a complete graph. Let $u, w \in V(G)$ and let P be a path from u to w, say $u = x_0, x_1, \ldots, x_m = w$. Then, since $x_0, x_2 \in N(x_1)$, $x_0x_2 \in E(G)$. Since $x_0, x_3 \in N(x_2)$, $x_0, x_3 \in E(G)$. Continuing in this manner, we find that $x_0x_m \in E(G)$, so u and w are adjacent. Sine u and w were chosen arbitrarily, this shows that G is complete.

5. Euler's formula says that in a connected, planar graph

$$n - q + r = 2,$$

where n is the number of vertices, q is the number of edges, and r is the number of regions in a planar representation of G. Find a similar formula for a disconnected graph with k connected, planar components. Justify your answer.

Let n_i, q_i, and r_i be the number of vertices, edges, and regions of the ith connected component of G. Then we have that

$$n = \sum_{i=1}^{k} n_i \quad \text{and} \quad q = \sum_{i=1}^{k} q_i.$$

However, the situation with the regions is slightly more complicated. In each connected component, the exterior region is counted in r_i. This means that

$$r = 1 + \sum_{i=1}^{k} (r_i - 1) = 1 - k + \sum_{i=1}^{k} r_i.$$

Therefore,

$$n - q + r = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} q_i + 1 - k + \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} (n_1 - q_i + r_i) + 1 - k = 2k + 1 - k = k + 1.$$

6. Write the word "True" next to each statement that is always true. Write "False" next to each statement that is not true or is only sometimes true.

______ If G is planar, then its compliment is planar.

False. For example, E_n is planar, but its compliment is K_5.

______ There is a connected graph with one vertex of odd degree that contains an Eulerian trail.

False. There is no such thing as a graph with only one vertex of odd degree.

______ If the degree sequence of G is 4,4,4,3,3, then G must be Hamiltonian.

True, since $3 = \delta(G) \geq n/2 = 2.5$.

______ A planar graph with 6 edges must have at least 4 vertices.

True. Since $q \leq 3n - 6$, $(q + 6)/3 \leq n$. If $q = 6$, then $n \geq 4$.

______ If r is a positive integer, then there is an r-regular graph that is planar.

False. If G is planar, then $\delta(G) \leq 5$.