1 Homotopy Category of Chain Complexes

1.1 Starting Definitions

**Definition.** We say that morphisms \( f, g \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(C_\ast, D_\ast) \) are chain homotopic, denoted by \( f \simeq g \), if there exists a family of morphisms \( s_n \in \text{Hom}_{\mathcal{A}}(C_n, D_{n+1}) \) indexed by \( n \in \mathbb{Z} \) so that

\[
    f_n - g_n = d_{D_{n+1}} \circ s_n + s_{n-1} \circ d_{C_n}
\]

for each \( n \in \mathbb{Z} \) where \( d_{D_{n+1}} \) and \( d_{C_n} \) are boundary maps of \( D_\ast \) and \( C_\ast \) respectively.

We say that a morphism \( f \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(C_\ast, D_\ast) \) is nullhomotopic if \( f \simeq 0 \).

**Note:** The diagram to have in mind when thinking about chain homotopy is the following:

\[
    \cdots \xrightarrow{d_{C_{n+1}}} C_{n+1} \xrightarrow{d_{C_n}} C_n \xrightarrow{d_{C_{n-1}}} \cdots \quad \text{with} \quad f_{n+1} \circ g_n = f_n \circ s_n + s_{n-1} \circ g_{n-1}
\]

\[
    \cdots \xrightarrow{d_{D_{n+1}}} D_{n+1} \xrightarrow{d_{D_n}} D_n \xrightarrow{d_{D_{n-1}}} \cdots
\]

This diagram is not commutative!

**Note:** Homotopic chain maps form an equivalence relation on \( \text{Hom}_{\mathcal{C}(\mathcal{A})}(C_\ast, D_\ast) \).

**Note:** If \( f \simeq g \) then \( f_\ast = g_\ast \) where \( f_\ast \) and \( g_\ast \) are the induced maps on homology (this is because \( f - g \) sends cycles to boundaries).

**Exercise:** Prove or disprove: If \( f \simeq g \) via a chain homotopy \( s_n : C_n \to D_{n+1} \) then \( (s_n)_{n \in \mathbb{Z}} : C_{\ast+1} \to D_\ast \) is a chain map.

**Proposition.** Assume \( f, g \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(C_\ast, D_\ast) \) with \( f \simeq g \) then \( v \circ f \circ u \simeq v \circ g \circ u \) for any \( u \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(B_\ast, C_\ast) \) and \( v \in \text{Hom}_{\mathcal{C}(\mathcal{A})}(D_\ast, E_\ast) \)
Proof. Assume that \( f \simeq g \) via \( (s_n)_{n \in \mathbb{Z}} \) and consider \( r_n := v_{n+1} \circ s_n \circ u_n \in \text{Hom}_A(B_n, E_{n+1}) \) indexed by \( n \in \mathbb{Z} \). We shall show that \( (r_n)_{n \in \mathbb{Z}} \) is a chain homotopy of \( v \circ f \circ u \) and \( v \circ g \circ u \):

\[
(v \circ f \circ u)_n - (v \circ g \circ u)_n = v_n \circ f_n \circ u_n - v_n \circ g_n \circ u_n
\]

(definition)

\[
v_n \circ (f_n - g_n) \circ u_n
\]

(\( A \) is additive)

\[
v_n \circ (d_{D_n + 1} \circ s_n + s_{n-1} \circ d_{C_n}) \circ u_n
\]

(\( f \simeq g \))

\[
v_n \circ d_{D_n + 1} \circ s_n \circ u_n + v_n \circ s_{n-1} \circ d_{C_n} \circ u_n
\]

(\( A \) is additive)

\[
d_{E_{n+1}} \circ v_{n+1} \circ s_n \circ u_n + v_n \circ s_{n-1} \circ u_{n-1} \circ d_{B_n}
\]

(u, v are chain maps)

\[
d_{E_{n+1}} \circ r_n + r_{n-1} \circ d_{B_n},
\]

(definition)

as desired. \( \square \)

**Definition.** We define the homotopy category of chain complexes, denoted by \( K(A) \), via

- \( \text{Obj}(K(A)) := \text{Obj}(C(A)) \)
- \( \text{Hom}_{K(A)}(C_*, D_*) := \text{Hom}_{C(A)}(C_*, D_*)/ \simeq \)
- \( [f] \circ [g] := [f \circ g] \).

where \( \simeq \) denotes the equivalence relation of chain homotopy.

**Note:** \( K(A) \) is actually a category.

The only thing to check is that the induced composition from \( C(A) \) is well-defined, but this follows from the above proposition.

## 2 Structure of \( K(A) \)

**Theorem.** \( K(A) \) is additive.

**Proof.** Earlier we saw that \( C(A) \) is an additive category, so we shall tacitly use this result.

(AD1) \( \text{Hom}_{C(A)}(C_*, D_*) \) is an abelian group, so \( \text{Hom}_{K(A)}(C_*, D_*) \) is a quotient of an abelian group. It follows that \( \text{Hom}_{K(A)}(C_*, D_*) \) is also an abelian group, as needed.

To see that we have a well-defined bilinear operation, notice that

\[
f \circ (u + v) = f \circ u + f \circ v \quad (C(A) \text{ is additive})
\]

\[
\simeq g \circ u + g \circ v \quad (\text{proposition})
\]

\[
= g \circ (u + v) \quad (C(A) \text{ is additive})
\]

The same proof works to show that \( (u + v) \circ g \) is well-defined.

(AD2) \( \text{Hom}_{K(A)}(0_{C(A)}, C_*) \) is a quotient of \( \text{Hom}_{C(A)}(0_{C(A)}, C_*) = \{0\} \) so \( \text{Hom}_{K(A)}(0_{C(A)}, C_*) = \{0\} \) as well. Using the same argument, \( \text{Hom}_{K(A)}(C_*, 0_{C(A)}) = \{0\} \) so we find that \( K(A) \) has a zero object which is the same as the zero object in \( C(A) \).
We shall show that \( \cdot \times_{C(A)} \cdot \) is the binary product in \( K(A) \). To this end, consider the following diagram in \( K(A) \):

\[
\begin{array}{ccc}
E_* & \xrightarrow{[\phi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\phi_D]} D_* \\
\downarrow{[\pi_C]} & & \downarrow{[\pi_D]} \\
C_* & \xrightarrow{\pi_C} & C_* \times_{C(A)} D_* \xrightarrow{\pi_D} D_*
\end{array}
\]

By choosing representatives of \( \phi_C, \phi_D, \pi_C, \) and \( \pi_D \) in \( C(A) \), we get a map \( f \) in \( C(A) \) so that the following diagram commutes in \( C(A) \):

\[
\begin{array}{ccc}
E_* & \xrightarrow{\phi_C} & C_* \times_{C(A)} D_* \xrightarrow{\phi_D} D_* \\
\downarrow{f} & & \downarrow{f} \\
C_* & \xrightarrow{\pi_C} & C_* \times_{C(A)} D_* \xrightarrow{\pi_D} D_*
\end{array}
\]

It follows that the following diagram commutes in \( K(A) \):

\[
\begin{array}{ccc}
E_* & \xrightarrow{[\phi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\phi_D]} D_* \\
\downarrow{[f]} & & \downarrow{[f]} \\
C_* & \xrightarrow{[\pi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\pi_D]} D_*
\end{array}
\]

We shall now show that the map above is unique. With this in mind, assume that we have two maps \( g, h \) so that the following diagram commutes in \( K(A) \):

\[
\begin{array}{ccc}
E_* & \xrightarrow{[\phi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\phi_D]} D_* \\
\downarrow{[g]} & & \downarrow{[h]} \\
C_* & \xrightarrow{[\pi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\pi_D]} D_*
\end{array}
\]

If we define \( [f] := [g] - [h] = [g - h] \) (using AD1) then we find that the following diagram commutes in \( K(A) \):

\[
\begin{array}{ccc}
E_* & \xrightarrow{[\phi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\phi_D]} D_* \\
\downarrow{[0]} & & \downarrow{[0]} \\
C_* & \xrightarrow{[\pi_C]} & C_* \times_{C(A)} D_* \xrightarrow{[\pi_D]} D_*
\end{array}
\]

It suffices to show that \( f \) is nullhomotopic.

With this in mind, since the above diagram commutes in \( K(A) \), we know that \( \pi_C \circ f \simeq 0 \) and \( \pi_D \circ f \simeq 0 \) via chain homotopies \( \{ s_n \}_{n \in \mathbb{Z}} \) and \( \{ s_D_n \}_{n \in \mathbb{Z}} \) respectively. By the above proposition, it follows that both \( \iota_{D_n} \circ \pi_{D_n} \circ f_n \) and \( \iota_{C_n} \circ \pi_{C_n} \circ f_n \) are nullhomotopic where
\( \iota_{C_n} : C_n \to C_n \times D_n \) is the canonical injection into the coproduct (the canonical injection exists and the coproduct equals the product since \( \mathcal{C}(A) \) is additive).

This tells us that the map
\[
\iota_{C_n} \circ \pi_{C_n} \circ f_n + \iota_{D_n} \circ \pi_{D_n} \circ f_n = \left( \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} \right) \circ f_n.
\]
is nullhomotopic. Therefore, it suffices to show that \( \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} = \text{Id}_{C_n \times D_n} \). Also, for ease of notation, define \( \phi_n := \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} \). Now, let us consider the following diagram in \( A \):

\[
\begin{array}{ccc}
C_n \times D_n & \xrightarrow{\iota_{C_n}} & C_n \\
\downarrow{\pi_{C_n}} & & \downarrow{\pi_{D_n}} \\
C_n & \xleftarrow{\phi_n} & C_n \times D_n & \xrightarrow{\pi_{D_n}} & D_n
\end{array}
\]

Since \( \pi_{C_n} \circ \iota_{C_n} = \text{Id}_{C_n} \) and \( \pi_{C_n} \circ \iota_{D_n} = 0 \), we find that
\[
\pi_{C_n} \circ \phi_n = \pi_{C_n} \circ \left( \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} \right)
= \pi_{C_n} \circ \iota_{C_n} \circ \pi_{C_n} + \pi_{C_n} \circ \iota_{D_n} \circ \pi_{D_n}
= \text{Id}_{C_n} \circ \pi_{C_n} + 0 \circ \pi_{D_n}
= \pi_{C_n}
\]
so the left triangle above commutes. By the same argument, the right triangle commutes, so we get that the diagram above is commutative. However, \( \text{Id}_{C_n \times D_n} \) makes the above diagram commute, so, by the universal property of products,
\[
\text{Id}_{C_n \times D_n} = \phi_n = \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n}
\]
as we set out to show.

Since \( \left( \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} \right) \circ f_n \) is nullhomotopic and \( \iota_{C_n} \circ \pi_{C_n} + \iota_{D_n} \circ \pi_{D_n} = \text{Id}_{C_n \times D_n} \), \( f_n \) is, itself, nullhomotopic, as desired. Hence, we have shown that the product map is unique, as desired.

We have shown that \( K(A) \) satisfies AD1, AD2, and AD3; so \( K(A) \) is an additive category, as desired. \( \square \)

### 3 Triangulated Categories

#### 3.1 Starting Definitions

The example above was supposed to act as motivation for why we need to introduce a different categorical structure. In particular, we shall show that \( K(A) \) is a triangulated category (which we will define later).

**Definition.** Let \( A \) and \( B \) be additive categories. We say a functor \( \Sigma : A \to B \) is additive if for every pair of objects \( X, Y \) in \( A \), the map
\[
\text{Hom}_A(X, Y) \xrightarrow{\Sigma} \text{Hom}_B(\Sigma(X), \Sigma(Y))
\]
is an abelian group morphism.

In addition, we say that an additive functor \( \Sigma : A \to A \) is an automorphism if there exists another functor \( \Sigma^{-1} : A \to A \) so that both \( \Sigma \circ \Sigma^{-1} \) and \( \Sigma^{-1} \circ \Sigma \) equal the identity functors.
Notation: From now on, assume that $\Sigma : \mathcal{A} \to \mathcal{A}$ is an additive, automorphism functor.

**Definition.** A triangle in $\mathcal{A}$ is a sequence of morphisms and objects in $\mathcal{A}$ of the form

$$X \overset{u}{\longrightarrow} Y \overset{v}{\longrightarrow} Z \overset{w}{\longrightarrow} \Sigma(X)$$

We will usually write a triangle as

$$Z \overset{w}{\longrightarrow} \overset{v}{\longleftarrow} \overset{u}{\longrightarrow} Y$$

**Definition.** A morphism of triangles is a triple $(f, g, h)$ of morphisms in $\mathcal{A}$ so that the following diagram is commutative in $\mathcal{A}$:

$$
\begin{array}{ccc}
X & \overset{u}{\longrightarrow} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \overset{u'}{\longrightarrow} & Y'
\end{array}
\begin{array}{ccc}
& \overset{v}{\longrightarrow} & \\
& \downarrow{h} & \\
& \overset{v'}{\longleftarrow} & \\
& \overset{w}{\longrightarrow} & \\
Z & \overset{w'}{\longrightarrow} & \Sigma(X')
\end{array}
$$

If each $f, g, h$ is an isomorphism in $\mathcal{A}$, we say that $(f, g, h)$ is an isomorphism of triangles.

**Note:** Since an additive category may not have short exact sequences, we can think of triangles as substitutes for short exact sequences.

**Definition.** A triangulated category is an additive category $\mathcal{A}$ together with an additive automorphism function $\Sigma$ (which we call the translation, shift, or suspension functor) and a collection of distinguished triangles $\{(u, v, w)\}$ that satisfy the following axioms:

(TR0) Any triangle isomorphic to a distinguished triangle is itself a distinguished triangle.

(TR1) For every object $X$ in $\mathcal{A}$, the triangle

$$
\begin{array}{ccc}
0 & \overset{0}{\longrightarrow} & 0 \\
\downarrow{0} & & \downarrow{0} \\
X & \overset{\text{Id}}{\longrightarrow} & X
\end{array}
$$

is a distinguished triangle.

(TR2) For every morphism $f : X \to Y$ in $\mathcal{A}$, there is a distinguished triangle of the form

$$
\begin{array}{ccc}
Z & \overset{f}{\longrightarrow} & Y \\
\downarrow{h} & & \downarrow{h} \\
X & \overset{g}{\longrightarrow} & Z
\end{array}
$$
(TR3) **Rotation Axiom:**

\[
\begin{array}{c}
X \\ u \\
\downarrow \quad v \\
Y \\
\downarrow \\
Z \\
w \\
\end{array}
\]

is a distinguished triangle if, and only if

\[
\begin{array}{c}
Y \\ \downarrow \quad v \\
\downarrow \\
Z \\
w \\
\end{array} \xleftarrow{-\Sigma(u)} \xrightarrow{\Sigma(X)} \\
\begin{array}{c}
X \\ u \\
\downarrow \quad v \\
\downarrow \\
Y \\
\downarrow \\
Z \\
w \\
\end{array}
\]

is a distinguished triangle.

(TR4) Given distinguished triangles

\[
\begin{array}{c}
X \\ u \\
\downarrow \quad v \\
Y \\
\downarrow \\
Z \\
w \\
\end{array} \quad \text{and} \quad \begin{array}{c}
X' \\ u' \\
\downarrow \quad v' \\
Y' \\
\downarrow \\
Z' \\
w' \\
\end{array}
\]

then every commutative diagram in \( A \) of the form:

\[
\begin{array}{c}
X \\ u \\
\downarrow \quad v \\
Y \\
\downarrow \quad g \\
Z \\
w \\
\downarrow \quad \Sigma(f) \\
\end{array} \xrightarrow{f} \xrightarrow{g} \xrightarrow{\Sigma(X)} \xrightarrow{\Sigma(f)} \]

\[
\begin{array}{c}
X' \\ u' \\
\downarrow \quad v' \\
Y' \\
\downarrow \\
Z' \\
w' \\
\end{array}
\]

can be completed to a morphism of triangles (not necessarily uniquely).

(TR5) **Octahedral Axiom:** Given the following distinguished triangles:

\[
\begin{array}{c}
X \\ u \\
\downarrow \quad v \\
Y \\
\downarrow \\
Z \\
w \\
\end{array} \quad , \quad \begin{array}{c}
X' \\ u' \\
\downarrow \quad v' \\
Y' \\
\downarrow \\
Z' \\
w' \\
\end{array} \quad , \quad \text{and} \quad \begin{array}{c}
X \\ v_u \\
\downarrow \quad \downarrow \\
Z \\
w \\
\end{array}
\]

there exists another distinguished triangle

\[
\begin{array}{c}
X' \\
\downarrow \quad \downarrow \\
Z' \\
w' \\
\end{array} \xleftarrow{X'} \xrightarrow{Y'}
\]
so that the following diagram commutes in $\mathcal{A}$:

$$
\begin{array}{ccccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z' & \xrightarrow{} & \Sigma(X) \\
\downarrow{\text{Id}_X} & & \downarrow{v} & & \downarrow{\text{Id}_{\Sigma(X)}} \\
X & \xrightarrow{vu} & Z & \xrightarrow{} & Y' & \xrightarrow{} & \Sigma(X) \\
\downarrow{u} & & \downarrow{\text{Id}_Z} & & \downarrow{\Sigma(u)} \\
Y & \xrightarrow{v} & Z & \xrightarrow{} & X' & \xrightarrow{} & \Sigma(Y) \\
\downarrow{} & & \downarrow{} & & \downarrow{\text{Id}_{X'}} \\
Z' & \xrightarrow{} & Y' & \xrightarrow{} & X' & \xrightarrow{} & \Sigma(Z')
\end{array}
$$

**Note:** TR0 is almost tautological: Just state that anything that is isomorphic to a distinguished triangle is itself a distinguished triangle.

**Note:** As stated above, triangles act as our substitute for short exact sequences. In fact, we get many results that are similar to important results in homological algebra:

For example, we can think of the Octahedral Axiom as a substitution for the Third Isomorphism Theorem. Since triangles are substitutes for short exact sequences, we can pretend that $Z' = Y/X$, $Y' = Z/X$, and $X' = Z/Y$. Then, we get that $X' = Y'/Z'$ so

$$
Z/Y = X' = Y'/Z' = (Z/X)/(Y/X)
$$

which is the Third Isomorphism Theorem!

By similar reasoning, we can think of TR2 as saying “cokernels” exist (and TR2 plus TR3 as “kernels” exist) and TR4 as the five lemma (TR 4 can actually be derived from the other axioms). There is also a version of the $3 \times 3$ lemma for triangulated categories.

With all these similarities, it seems like there should be a homology theory for triangulated categories (and there is!). This theory comes from cohomological functors:

**Definition.** An additive contravariant functor $F : \mathcal{A} \to \mathcal{C}$ from a triangulated category $\mathcal{A}$ to an abelian category $\mathcal{C}$ is called a cohomological functor if for any distinguished triangle:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & \xrightarrow{} & \Sigma(X)
\end{array}
$$

the sequence

$$
\begin{array}{ccc}
F(Z) & \xrightarrow{F(g)} & F(Y) \\
& \xrightarrow{F(f)} & F(X)
\end{array}
$$

is exact in $\mathcal{C}$.

Notice that any cohomological functor on a distinguished triangle gives rise to a long exact sequence in $\mathcal{C}$ (which replicates the long exact sequence for the usual homology).

The prototypical example of a cohomological functor is $\text{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \text{AblGrP}$ where $X$ is some object in $\mathcal{A}$. 

7
Note: The name of the Octahedral Axiom comes from redrawing the commutative diagram as:

\[ \begin{array}{ccc}
Y' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Z & \leftarrow & X \\
\uparrow & & \uparrow \\
X & \rightarrow & Z \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y'
\end{array} \]

Note: It is an open question where there exists a category that satisfies every axiom except the Octahedral axiom.

Note: We do not actually need all of the axioms above. TR4 and one direction of TR3 can be deduced from the others axioms.

3.2 \( \mathbf{K} (\mathcal{A}) \) is a Triangulated Category

As we saw above, \( \mathbf{K} (\mathcal{A}) \) is not necessarily an abelian category (even if \( \mathcal{A} \) is abelian). However, \( \mathbf{K} (\mathcal{A}) \) has the structure of a triangulated category. In this section, we shall describe the structure and then prove that it satisfies the axioms above.

In particular, to show that \( \mathbf{K} (\mathcal{A}) \) is triangulated, we must find a suitable translation functor and then find a suitable collection of distinguished triangles. This gives us our first definition of this subsection:

**Definition.** We define the translation functor \([1] : \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{C}(\mathcal{A})\) by shifting each complex one degree to the left. Specifically, for \( C_* = (C_n, d_{C_n})_{n \in \mathbb{Z}} \), we say
\[
C_*[1] := (C_{n-1}, -d_{C_{n-1}})_{n \in \mathbb{Z}}
\]
and for \( f = (f_n)_{n \in \mathbb{Z}} \in \text{Hom}_{\mathbf{C}(\mathcal{A})}(C_*, D_*) \), we say
\[
f[1] := (f_{n-1})_{n \in \mathbb{Z}}.
\]

Note: \([1]\) is an additive automorphism functor.

Also, while the negative in the boundary map may seem extraneous, it is essential to make the triangulated category axioms work out.

Now that we have our translation functor, we want to define the triangles in our category.

**Definition.** Given a chain map \( f \in \text{Hom}_{\mathbf{C}(\mathcal{A})}(C_*, D_*) \), we define the mapping cone \( M(f) \) to be the chain complex in \( \mathbf{C}(\mathcal{A}) \) where
\[
M(f)_n = C_{n-1} \times D_n \quad \text{and} \quad d_{M(f)} = \begin{bmatrix} -d_{C_{n-1}} & 0 \\ f_{n-1} & d_{D_n} \end{bmatrix} = \begin{bmatrix} d_{C[1]_n} & 0 \\ f[1]_n & d_{D_n} \end{bmatrix}
\]
Note: Notice that there are some obvious maps associated with $M(f)$ in $\mathbf{C}(\mathcal{A})$. Specifically, we have chain maps:

$$\alpha(f) : D_* \to M(f), \quad \alpha(f)_n := (0, \text{Id}_{D_n})$$

and

$$\beta(f) : M(f) \to C_*[1], \quad \beta(f)_n := (\text{Id}_{C_{n-1}}, 0)$$

**Definition.** A sequence in $\mathbf{K}(\mathcal{A})$ of the form

$$C_* \xrightarrow{f} D_* \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} C_*[1]$$

is called a standard triangle where we are identifying $f$, $\alpha(f)$, $\beta(f)$ with their chain homotopic equivalence classes.

We will say that a triangle in $\mathbf{K}(\mathcal{A})$ is distinguished if it is isomorphic to a standard triangle.

**Proposition.** Let $\mathcal{A}$ be an additive category, then $\mathbf{K}(\mathcal{A})$ with distinguished triangles given above and translation functor $[1]$ is a triangulated category.

**Proof.**

(TR0) This follows from our definition of distinguished triangles.

(TR1) Notice that

$$M(\text{Id}_{C_*}) = \left( C_* \times C_{*-1}, \begin{bmatrix} -d_{C_{*-1}} & 0 \\ \text{Id}_{C_{*-1}} & d_{C_*} \end{bmatrix} \right),$$

and

$$\begin{bmatrix} -d_{C_*} & 0 \\ \text{Id}_{C_*} & d_{C_{*-1}} \end{bmatrix} \begin{bmatrix} 0 & \text{Id}_{C_*} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \text{Id}_{C_{*-1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_{C_{*-1}} & 0 \\ \text{Id}_{C_{*-1}} & d_{C_*} \end{bmatrix} = \begin{bmatrix} 0 & -d_{C_*} \\ \text{Id}_{C_*} & 0 \end{bmatrix} + \begin{bmatrix} \text{Id}_{C_{*-1}} & d_{C_*} \\ 0 & 0 \end{bmatrix} = \text{Id}_{M(\text{Id}_{C_*})},$$

so, by definition, the identity map on $M(\text{Id}_{C_*})$ is nullhomotopic. In other words, the triangle

$$\begin{tikzcd}
\text{Id}_{C_*} & C_* \\
M(\text{Id}_{C_*}) & C_*
\arrow{rr}{\text{Id}_{C_*}} & & \arrow{ll}{C_*}
\end{tikzcd}$$

is isomorphic to the triangle

$$\begin{tikzcd}
0 \\
\text{Id}_{C_*} & C_* & \text{Id}_{C_*}
\arrow{rr}{0} & & \arrow{ll}{C_*}
\end{tikzcd}$$

Thus, by definition, the triangle directly above is a distinguished triangle, as needed.

(TR2) This follows from our definition of distinguished triangles.

(TR3) By definition of a distinguished triangle, it suffices to prove the rotation axiom for a standard triangle. With this in mind, consider the standard triangle

$$C_* \xrightarrow{f} D_* \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} C_*[1].$$
We shall show that the rotated triangle
\[
D_* \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} C_*[1] \xrightarrow{-f[1]} D_*[1]
\]
is isomorphic in $K(A)$ to the standard triangle for $\alpha(f)$:
\[
D_* \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(f)} M(\alpha(f)) \xrightarrow{\beta(f)} D_*[1].
\]
With this in mind, define $\phi : C_*[1] \to M(\alpha(f))$ via
\[
\phi_n := (-f_{n-1}, \text{Id}_{C_{n-1}}, 0)
\]
and $\psi : M(\alpha(f)) \to C_*[1]$ via
\[
\psi_n = (0, \text{Id}_{C_{n-1}}, 0).
\]
Hence, we have the following morphisms of triangles in $K(A)$:
\[
\begin{array}{cccccccccccc}
D_* & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & C_*[1] & \xrightarrow{-f[1]} & D_*[1] \\
\downarrow & & \downarrow & & \phi & & \downarrow & & \text{Id}_{D_*[1]} \\
D_* & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\alpha(f)} & M(\alpha(f)) & \xrightarrow{\beta(f)} & D_*[1] \\
\end{array}
\]
We know that $(\text{Id}_{D_*}, \text{Id}_{M(f)}, \phi)$ is actually a morphism of triangles because, by definition, $\beta(\alpha(f)) \circ \phi = -f[1]$, and $\phi \circ \beta(f) \simeq \alpha(\alpha(f))$ via
\[
\begin{bmatrix}
0 & -\text{Id}_{D_n} \\
0 & 0 \\
0 & 0
\end{bmatrix} : C_{n-1} \times D_n = M(f)_n \to M(\alpha(f))_{n+1} = D_n \times C_n \times D_{n+1}
\]
By similar reasoning, $(\text{Id}_{D_*}, \text{Id}_{M(f)}, \psi)$ is also a morphism of triangles.
It remains to show that $\phi$ is an isomorphism via $\psi$. However, by definition $\psi \circ \phi = \text{Id}_{C_*[1]}$ and $\phi \circ \psi \simeq \text{Id}_{M(\alpha(f))}$ via the map:
\[
\begin{bmatrix}
0 & 0 & -\text{Id}_{D_n} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} : D_{n-1} \times C_{n-1} \times D_n = M(\alpha(f))_n \to M(\alpha(f))_{n+1} = D_n \times C_n \times D_{n+1}.
\]
Hence, we have that $(\text{Id}_{D_*}, \text{Id}_{M(f)}, \phi)$ is an isomorphism of triangles so by TR0, $D_* \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} C_*[1] \xrightarrow{-f[1]} D_*[1]$ is distinguished, as needed.
The other direction follows from definition.

(TR4) Once again, it suffices to show that this axiom holds for standard triangles. With this in mind, assume we have the following diagram where the left square commutes in $K(A)$:
\[
\begin{array}{cccccccccccc}
C_* & \xrightarrow{u} & D_* & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(u)} & C_*[1] \\
\downarrow & & \downarrow & & \downarrow & & \text{Id}_{C_*[1]} \\
C'_* & \xrightarrow{u'} & D'_* & \xrightarrow{\alpha(u')} & M(u') & \xrightarrow{\beta(u')} & C'_*[1]
\end{array}
\]
That means, $u \circ g_n \simeq f \circ u'$ via maps $s_n : C_n \to D'_{n+1}$. Now, define $h_n : M(u)_n \to M(u')_n$ via

$$h_n = \begin{bmatrix} f_{n-1} & 0 \\ s_{n-1} & g_n \end{bmatrix} : C_{n-1} \times D_n = M(u)_n \to M(u')_n = C'_{n-1} \times D'_n$$

If you just write it out (and using that $s_n$ is a chain homotopy), you can see that $h_n$ is a boundary map. Moreover, $(f, g, h)$ is a morphism of triangles because, by definition, $h \circ \alpha(u) = \alpha(u') \circ g$ and $\beta(u') \circ h = f[1] \circ \beta(u)$, as desired.

(TR5) Just as above, we can assume that all triangles are standard triangles. Thus, assume we have the following diagram that commutes in each $1 \times 1$ square:

\[
\begin{array}{ccc}
C_n & \xrightarrow{u} & D_n \\
\downarrow{\text{Id}_{C_n}} & & \downarrow{\text{Id}_{D_n}} \\
C_n & \xrightarrow{vu} & Z \\
\downarrow{u} & & \downarrow{\text{Id}_{E_n}} \\
D_n & \xrightarrow{v} & E_n \\
\downarrow{\alpha(u)} & & \downarrow{\alpha(v)} \\
M(u) & \xrightarrow{\alpha(vu)} & M(v) \\
\downarrow{\text{Id}_{M(v)}} & & \downarrow{\text{Id}_{M(v)}} \\
M(u) & \xrightarrow{\alpha(u)} & M(v) \\
\end{array}
\]

With that diagram in mind, define $f : M(u) \to M(vu)$ via

$$f_n = \begin{bmatrix} \text{Id}_{C_n-1} & 0 \\ 0 & v_n \end{bmatrix},$$

$g : M(vu) \to M(v)$ via

$$g_n = \begin{bmatrix} u_{n-1} & 0 \\ 0 & \text{Id}_{E_n} \end{bmatrix},$$

and $h : M(v) \to M(u)[1]$ as $h = \alpha(u)[1] \circ \beta(v)$. By writing everything out, we see that the following diagram commutes in $K(A)$:

\[
\begin{array}{ccc}
C_n & \xrightarrow{u} & D_n \\
\downarrow{\text{Id}_{C_n}} & & \downarrow{\text{Id}_{D_n}} \\
C_n & \xrightarrow{vu} & Z \\
\downarrow{u} & & \downarrow{\text{Id}_{E_n}} \\
D_n & \xrightarrow{v} & E_n \\
\downarrow{\alpha(u)} & & \downarrow{\alpha(v)} \\
M(u) & \xrightarrow{\alpha(vu)} & M(v) \\
\downarrow{\text{Id}_{M(v)}} & & \downarrow{\text{Id}_{M(v)}} \\
M(u) & \xrightarrow{\alpha(u)} & M(v) \\
\end{array}
\]
We must show that
\[ M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} M(u)[1] \]
is a distinguished triangle. Therefore, it suffices to show that the above triangle is isomorphic to the triangle:
\[ M(u) \xrightarrow{f} M(vu) \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} M(u)[1] \]

Therefore, construct \( \sigma : M(v) \to M(f) \) and \( \tau : M(f) \to M(v) \) via
\[
\sigma_n := \begin{bmatrix} 0 & 0 \\ Id_{D_{n-1}} & 0 \\ 0 & 0 \\ 0 & Id_{E_n} \end{bmatrix} : D_{n-1} \times E_n \to C_{n-2} \times D_{n-1} \times C_{n-1} \times E_n \\
and
\tau_n = \begin{bmatrix} 0 & Id_{D_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & Id_{C_n} \end{bmatrix} : C_{n-2} \times D_{n-1} \times C_{n-1} \times E_n \to D_{n-1} \times E_n
\]

Then, you just need to confirm that the following diagram commutes in \( K(A) \):

\[
\begin{array}{cccccc}
M(u) & \xrightarrow{f} & M(vu) & \xrightarrow{g} & M(v) & \xrightarrow{h} & M(u)[1] \\
\downarrow \Id_{M(u)} & & \downarrow \Id_{M(vu)} & & \downarrow \Id_{M(v)} & & \downarrow \Id_{M(u)[1]} \\
M(u) & \xrightarrow{f} & M(vu) & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & M(u)[1]
\end{array}
\]

Notice, by definition, that \( \tau \circ \alpha(f) = g \) and \( \beta(f) \circ \sigma = h \). Moreover, \( \sigma \circ g \simeq \alpha(f) \) via
\[
\begin{bmatrix} Id_{C_{n-1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : C_{n-1} \times E_n \to C_{n-1} \times D_n \times C_n \times E_{n+1}
\]

and, similarly, \( h \circ \tau \simeq \beta(f) \) via
\[
\begin{bmatrix} 0 & 0 & Id_{C_{n-1}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : C_{n-2} \times D_{n-1} \times C_{n-1} \times E_n \to C_{n-1} \times D_n.
\]

Therefore, the above diagram commutes.

It remains to show that we actually have an isomorphism of triangles. Following from definition, we have that \( \tau \circ \sigma = \Id_{M(v)} \). Then, we find that \( \sigma \circ \tau \simeq \Id_{M(f)} \) via
\[
\begin{bmatrix} 0 & 0 & -Id_{C_{n-1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : C_{n-2} \times D_{n-1} \times C_{n-1} \times E_n \to C_{n-1} \times D_n \times C_n \times E_{n+1}.
\]

Hence, \( \sigma \) is an isomorphism, so we find that
\[ M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} M(u)[1] \]
is a distinguished triangle, as desired. \( \square \)
3.3 Other Examples of Triangulated Categories

**Example:** If $\mathcal{A}$ is additive then any bounded version of $K(\mathcal{A})$ is triangulated (with the same triangulated structure as $K(\mathcal{A})$). In particular, the homotopy category of chain complexes bounded below, chain complexes bounded above, or chain complexes bounded both above and below.

**Example:** Vector Spaces over a fixed field form a triangulated category where the translation functor is the identity and a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ is distinguished if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y$$

is an exact sequence as vector space.

**Example:** If $\mathcal{A}$ is an abelian category then the derived category $D(\mathcal{A})$ is triangulated with the structure that we will learn in a different seminar. More generally, under special conditions, the localization of a triangulated category is also triangulated.

**Example:** Consider the stable homotopy category where objects are spectra with spectra morphisms. This category is triangulated where the translation functor is suspension and the distinguished triangles are cofibration sequences.

3.4 Where do Triangulated Categories Fit?

I think it is a natural question to ask where triangulated categories fit into other structures on categories. The following proposition gives us a slightly clearer picture:

**Proposition.** Let $\mathcal{A}$ be a triangulated, abelian category. Then $\mathcal{A}$ is semisimple.

**Definition.** An abelian category is semisimple if every exact sequence splits.

**Example:** If $R$ is a semisimple ring then $R$-Mod is semisimple. In particular, vector spaces over a fixed field is semisimple. Also, $\textbf{AbGrp}$ is not semisimple.

**Proof.** Let

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

be a short exact sequence in a triangulated, abelian category $\mathcal{A}$. By TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{u} V \xrightarrow{u} \Sigma(X)$$

Now, by TR1, we get the following commutative diagram:

$$
\begin{array}{c}
X & \xrightarrow{f} & Y & \xrightarrow{u} & V & \xrightarrow{u} & \Sigma(X) \\
| & | & | & | & | & | & | \\
\downarrow f & \downarrow \text{Id}_Y & \downarrow \text{Id}_Y & & \downarrow \Sigma(f) & & \\
Y & \xrightarrow{u} & 0 & \xrightarrow{u} & \Sigma(Y) \\
\end{array}
$$

13
By TR4, we can complete the above diagram to a morphism of triangles. In particular, $\Sigma(f) \circ u = 0$ so

$$0 = \Sigma(f) \circ u = \Sigma(f) \circ \Sigma^{-1}(u) = \Sigma(f \circ \Sigma^{-1}(u)).$$

Since $\Sigma$ is an automorphism functor, $f \circ \Sigma^{-1}(u) = 0$. Moreover, since

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

is a short exact sequence, $f$ is monic, so $\Sigma^{-1}(u) = 0$. Because $\Sigma$ is an automorphism functor, $u = 0$. That means, we can rewrite our earlier distinguished triangle as

$$X \xrightarrow{f} Y \xrightarrow{v} V \xrightarrow{0} \Sigma(X)$$

Therefore, the following diagram commutes in $A$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \text{Id}_X & & \downarrow \text{Id}_Y \\
X & \xrightarrow{\text{Id}_X \circ f} & X
\end{array}
\quad
\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Sigma(X)
\end{array}
\quad
\begin{array}{ccc}
\Sigma(X) & \rightarrow & 0 \\
\downarrow \text{Id}_{\Sigma(X)} & & \downarrow \\
\Sigma(X) & \rightarrow & 0
\end{array}
$$

Thus, by TR4 and TR3, we can complete the above diagram to a morphism of triangles. In particular, we get a map $\tilde{f} : Y \rightarrow X$ so that $\bar{f} \circ f = \text{Id}_X$ which means that the short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

splits in $A$, as desired. 

**Example:** $K(\text{AblGrp})$ is not abelian.

We proceed by contradiction, so assume that $K(\text{AblGrp})$ is abelian category. We saw above that $K(A)$ is triangulated for any additive category $A$. In particular, $K(\text{AblGrp})$ is triangulated, so by the proposition above, every short exact sequence in $K(\text{AblGrp})$ splits. With this in mind, consider the following short exact sequence in $K(\text{AblGrp})$:

$$
\begin{array}{ccccccccc}
\cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & 0 & \rightarrow & \mathbb{Z}/2\mathbb{Z} & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & 0 & \rightarrow & \mathbb{Z}/4\mathbb{Z} & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & 0 & \rightarrow & \mathbb{Z}/2\mathbb{Z} & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots
\end{array}
$$

where $f(0) = 0, f(1) = 2$ and $g(0) = 0, g(1) = 3, g(2) = 0, g(3) = 3$. By assumption, the above exact sequence splits, so, in particular, we have a map $\tilde{f} : \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ so that $\tilde{f} \circ f \simeq \text{Id}_{\mathbb{Z}/2\mathbb{Z}}$. However, notice that all chain complexes above are concentrated in degree zero, so any chain
homotopy must be the zero map. It follows that \( \tilde{f} \circ f = \text{Id}_{\mathbb{Z}/2\mathbb{Z}} \) so the short exact sequence

\[
0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0
\]

splits in \( \text{AblGrp} \). However, we know that the above short exact sequence does not split in \( \text{AblGrp} \) because \( \mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), a contradiction! Hence, the category \( K(\text{AblGrp}) \) is not abelian, as we set out to show.

This example gives us the following picture:

\[
\begin{array}{c}
\text{Abelian} \bigcap \text{Triangulated} \\
\bigcap \\
\text{Semisimple} \\
\text{AblGrp} \bigcap K(\text{AblGrp}) \\
\text{Abelian} \neq \text{Triangulated} \\
\text{Grp} \bigcap K(\text{AblGrp}) \\
\text{Additive} \\
\end{array}
\]

**Exercise:** Prove or disprove: Every semisimple category is triangulated.