ELLIPTIC COHOMOLOGY: A HISTORICAL OVERVIEW

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The goal of this overview is to introduce concepts which underlie elliptic cohomology and reappear in the construction of *tmf*. We begin by defining complex-oriented cohomology theories and looking at the two special cases of complex cobordism and K-theory. We then see that a complex orientation of a cohomology theory naturally leads to a formal group law. Furthermore, Quillen's theorem states that the universal complex-oriented theory (complex cobordism) encodes the universal formal group law. This implies that complex genera, or homomorphisms from the complex cobordism ring to a ring R, are equivalent to formal group laws over R. The group structure on an elliptic curve naturally leads to the notion of an elliptic genus. Finally, we use the Landweber exact functor theorem to produce an elliptic cohomology theory whose formal group law is given by the universal elliptic genus.

Elliptic cohomology was introduced by Landweber, Ravenel, and Stong in the mid-1980's as a cohomological refinement of elliptic genera. The notion of elliptic genera had previously been invented by Ochanine to address conjectured rigidity and vanishing theorems for certain genera on manifolds admitting non-trivial group actions. Witten played an important role in this process by using intuition from string theory to form many of these conjectures. He subsequently interpreted the elliptic genus as the signature of the free loop space of a spin manifold, beginning a long and interesting interaction between theoretical physics and algebraic topology that is still active today. While we don't have the space to adequately tell this story, there are already several excellent references: the introductory article in [Lan] gives the history of elliptic genera and elliptic cohomology, [Seg] explains how they should be related to more geometric objects, and [Hop] summarizes important properties of *tmf*. Finally, both [Lur] and [Goe] give a detailed survey of elliptic cohomology and *tmf* from the more modern perspective of derived algebraic geometry.

1. Complex-oriented cohomology theories

A generalized cohomology theory E is a functor from (some subcategory of) topological spaces to the category of abelian groups. This functor must satisfy all the Eilenberg–Steenrod axioms except for the dimension axiom, which states the cohomology of a point is only non-trivial in degree 0. Any cohomology theory is represented by a spectrum which we also call E, and from a spectrum the reduced homology and cohomology groups of a finite CW complex X are given by

$$\widetilde{E}_n(X) = \lim_{k\to\infty} \pi_{n+k}(X \wedge E_k),$$
$$\widetilde{E}^n(X) = \lim_{k\to\infty} [\Sigma^k X, E_{n+k}].$$

The coefficient groups are abbreviated by $E^* = E^*(\mathrm{pt})$ and $E_* = E_*(\mathrm{pt})$, and they are naturally related by $\pi_* E = E_* \cong E^{-*}$. We restrict to theories with a graded commutative ring structure $E^i(X) \times E^j(X) \to E^{i+j}(X)$ analogous to the cup product in ordinary cohomology. They are known as multiplicative cohomology theories and are represented by ring spectra.

Example 1.1 (Cobordism). A smooth closed (compact with no boundary) manifold M is said to be null-bordant if there exists a compact manifold W whose boundary is M. A singular manifold (M, f) in X, where $f : M \to X$ is a continuous map, is null-bordant if there exists a singular manifold (W, F) with boundary (M, f). The n-th unoriented bordism group of X, denoted by $\Omega_n^O(X)$, is the

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set of smooth closed singular n-manifolds in X modulo null-bordism; the group structure is given by the disjoint union of manifolds.

Let $\{G_k\}$ be a sequence of topological groups with representations $\{G_k \xrightarrow{\rho_k} O(k)\}$ which are compatible with the inclusion maps. We define a G-structure on M as a stable lift of the structure groups to G_k for the stable normal bundle ν_M. Suppose a manifold W with $\partial W = M$ has a G-structure on ν_W that extends to the G-structure on ν_M. This is considered a null-bordism of M as a G-manifold. The abelian group $\Omega_n^G(X)$ is then defined as before; it is the set of smooth closed singular n-manifolds on X with G-structure on ν, modulo null-bordism. Up to homotopy, G-structures on the stable tangent bundle and stable normal bundle are equivalent; we later use this fact in geometric constructions.

The functors Ω_*^G are examples of generalized homology theories, and the Pontryagin–Thom construction shows they are represented by the Thom spectra $MG = \{MG_k\} = \{Th(\rho_k^* \xi_k)\}$. Here, $\xi_k \to BO(k)$ is the universal k-dimensional vector bundle ($\xi_k = EO(k) \times_{O(k)} \mathbb{R}^k$), and for any vector bundle $V \to X$ the Thom space $Th(V)$ is defined as the unit disc bundle modulo the unit sphere bundle $D(V)/S(V)$. Particularly common examples of G-bordism include oriented bordism, spin bordism, and complex bordism, corresponding to the groups $SO(k)$, $Spin(k)$, and $U(k)$, respectively. Bordism classes in these examples have an orientation, spin structure, or complex structure on the manifold's stable normal bundle (or stable tangent bundle).

The spectrum MG defines a generalized cohomology theory known as G-cobordism. It is also a multiplicative cohomology theory (assuming there are maps $G_{k_1} \times G_{k_2} \to G_{k_1+k_2}$ compatible with the orthogonal representations). The coefficient ring of MG is simply the bordism ring of manifolds with stable G-structure,

$$MG^{-*}(\mathrm{pt}) \cong MG_*(\mathrm{pt}) = \Omega_*^G,$$

and the product structure is induced by the product of manifolds. Of particular interest to us will be oriented cobordism and complex cobordism. The first coefficient calculation is due to Thom, and the second is from Thom, Milnor, and Novikov:

$$\begin{aligned} MSO^* \otimes \mathbb{Q} &\cong \mathbb{Q}[[\mathbb{CP}^2], [\mathbb{CP}^4], \ldots] \\ MU^* &\cong \mathbb{Z}[a_1, a_2, \ldots]; \quad |a_i| = -2i. \end{aligned} \tag{1}$$

Rationally, $MU^* \otimes \mathbb{Q}$ is generated by the complex projective spaces $\mathbb{CP}^i$ for $i \geq 1$. The book [Sto] is an excellent source of further information on cobordism.

Example 1.2 (Complex K-theory). Isomorphism classes of complex vector bundles over a space X form an abelian monoid via the direct sum $\oplus$ operation. Formally adjoining inverses gives the associated Grothendieck group known as $K(X)$ or $K^0(X)$; elements in $K(X)$ are formal differences of vector bundles up to isomorphism. The reduced group $\widetilde{K}^0(X)$ is naturally isomorphic to $[X, \mathbb{Z} \times BU]$, and Bott periodicity gives a homotopy equivalence $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Therefore, we can extend $\mathbb{Z} \times BU$ to an Ω-spectrum known as K, where

$$\begin{aligned} K_{2n} &= \mathbb{Z} \times BU, \\ K_{2n+1} &= \Omega(\mathbb{Z} \times BU) \simeq U. \end{aligned}$$

This defines the multiplicative cohomology theory known as (complex) K-theory, with ring structure induced by the tensor product of vector bundles. A straightforward evaluation shows that the coefficients $\pi_* K$ are

$$\begin{aligned} K^{2n}(\mathrm{pt}) &\cong \pi_0(\mathbb{Z} \times BU) = \mathbb{Z}, \\ K^{2n+1}(\mathrm{pt}) &\cong \pi_0(U) = 0. \end{aligned}$$

Furthermore, Bott periodicity is manifested in K-theory by the Bott class $\beta = [\xi] - 1 \in \widetilde{K}(S^2) \cong K^{-2}(\mathrm{pt})$, where $\xi \to S^2$ is the Hopf bundle and 1 is the isomorphism class of the trivial line bundle. The class β is invertible in K^*, and multiplication by β and β^{-1} induces the periodicity in general rings $K^*(X)$.

The periodicity in K-theory turns out to be a very convenient property, and it motivates the following definition.

Definition 1.3. A multiplicative cohomology theory E is *even periodic* if $E^i(\mathrm{pt}) = 0$ whenever i is odd and there exists $\beta \in E^{-2}(\mathrm{pt})$ such that β is invertible in $E^*(\mathrm{pt})$.

The existence of $\beta^{-1} \in E^2(\mathrm{pt})$ implies that for general X there are natural isomorphisms

$$E^{*+2}(X) \underset{\cdot\beta^{-1}}{\overset{\cdot\beta}{\rightleftarrows}} E^*(X)\ (\cong).$$

given by multiplication with β and β^{-1}, so E is periodic with period 2.

A number of cohomology theories, such as ordinary cohomology, are even (i.e. $E^{\mathrm{odd}}(\mathrm{pt}) = 0$) but not periodic. Given an arbitrary even cohomology theory, we can create an even periodic theory A by defining

$$A^n(X) := \prod_{k \in \mathbb{Z}} E^{n+2k}(X).$$

For example, if we perform this construction on ordinary cohomology with coefficients in a ring R, we obtain a theory known as periodic ordinary cohomology. The coefficients of MU in (1) show that MU also is even but not periodic. We define *periodic complex cobordism* MP by

$$MP^n(X) := \prod_{k \in \mathbb{Z}} MU^{n+2k}(X),$$

Letting $|\beta| = -2$, we could equivalently define $MP^n(X) \subset MU^*(X)[\![\beta, \beta^{-1}]\!]$ as formal series which are homogeneous of degree n.

Definition 1.4. In E-cohomology, a Thom class for the vector bundle $V \to X$ (with $\dim_{\mathbb{R}} V = n$) is a class $\mathcal{U}_V \in \widetilde{E}^n(Th(V))$ such that for each $x \in X$ there exists $\varphi_x : \mathbb{R}^n \to V_x$ so that $\mathcal{U}_V \mapsto 1$ under the following composition:

$$\begin{array}{ccccccc} \widetilde{E}^n(Th(V)) & \longrightarrow & \widetilde{E}^n(Th(V_x)) & \xrightarrow[\varphi_x^*]{\cong} & \widetilde{E}^n(S^n) & \xrightarrow{\cong} & E^0(\mathrm{pt}) \\ \mathcal{U}_V & \longmapsto & & & & & 1 \end{array}$$

Thom classes give rise to Thom isomorphisms

$$E^*(X) \xrightarrow{\cdot \mathcal{U}_V} \widetilde{E}^{*+n}(Th(V)).$$

The existence of Thom isomorphisms allows one to construct pushforward maps in cohomology theories, which in turn gives important invariants generalizing the Euler class. Ordinary cohomology with $\mathbb{Z}/2$ coefficients admits Thom classes for all vector bundles, but only oriented bundles have Thom classes in $H^*(-;\mathbb{Z})$. In general, we would like to functorially define Thom classes compatible with the $\oplus$ operation. Such a choice for vector bundles with lifts of the structure group to G_k is called a G-orientation of the cohomology theory E, and E is said to be G-orientable if there exists such an orientation. A specific orientation will be given by universal Thom classes in $\widetilde{E}^n(MG_n)$ and is equivalent (at least up to homotopy) to a map of ring spectra $MG \to E$. We will mostly be concerned with complex orientable theories, and summarizing the above discussion gives the following definition.

Definition 1.5. A complex orientation of E is a natural, multiplicative, collection of Thom classes $\mathcal{U}_V \in \widetilde{E}^{2n}(Th(V))$ for all complex vector bundles $V \to X$, where $\dim_{\mathbb{C}} V = n$. More explicitly, these classes must satisfy

- $f^*(\mathcal{U}_V) = \mathcal{U}_{f^*V}$ for $f : Y \to X$,
- $\mathcal{U}_{V_1 \oplus V_2} = \mathcal{U}_{V_1} \cdot \mathcal{U}_{V_2}$,

- For any $x \in X$, the class $\mathcal{U}_V$ maps to 1 under the composition[1]

$$\widetilde{E}^{2n}(Th(V)) \to \widetilde{E}^{2n}(Th(V_x)) \xrightarrow{\cong} \widetilde{E}^{2n}(S^{2n}) \xrightarrow{\cong} E^0(\mathrm{pt}).$$

Given a complex orientation, we can define Chern classes in the cohomology theory E. Because the zero-section $\mathbb{CP}^\infty \to \xi$ induces a homotopy equivalence $\mathbb{CP}^\infty \xrightarrow{\sim} Th(\xi)$, the universal Thom class for line bundles is naturally a class $c_1 \in \widetilde{E}^2(\mathbb{CP}^\infty)$, and it plays the role of the universal first Chern class. If one computes $E^*(\mathbb{CP}^\infty)$, the existence of c_1 implies the Atiyah–Hirzebruch spectral sequence must collapse at the E_2 page. This implies the first part of the following theorem.

Theorem 1.6. *A complex orientation of E determines an isomorphism*

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![c_1]\!],$$

and such an isomorphism is equivalent to a complex orientation. Furthermore, any even periodic theory is complex orientable.

In addition to the above proposition, the splitting principle carries over, and the class c_1 uniquely determines isomorphisms

$$\begin{aligned} E^*(BU(n)) &\cong (E^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty))^{\Sigma_n} \cong \left(E^*(\mathrm{pt})[\![x_1, \cdots, x_n]\!]\right)^{\Sigma_n} \\ &\cong E^*(\mathrm{pt})[\![c_1, \cdots, c_n]\!], \end{aligned}$$

where $c_k \in E^{2k}(BU(n))$ is the k-th elementary symmetric polynomial in the variables x_i. This gives us a theory of Chern classes analogous to the one in ordinary cohomology.

2. Formal group laws and genera

A complex orientation of E determines Chern classes for complex vector bundles. As in ordinary cohomology, the Chern classes satisfy the properties of naturality and additivity. In ordinary cohomology, the first Chern class of a product of line bundles is given by

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

For a general complex-oriented cohomology theory, this relation no longer holds and leads to an interesting structure.

The universal tensor product is classified by

$$\begin{array}{ccc} \xi \otimes \xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty & \longrightarrow & \mathbb{CP}^\infty. \end{array}$$

The induced map in cohomology,

$$\begin{aligned} E^*(\mathbb{CP}^\infty) \otimes E^*(\mathbb{CP}^\infty) &\longleftarrow E^*(\mathbb{CP}^\infty) \\ F(x_1, x_2) &\longleftarrow\!\shortmid c_1 \end{aligned}$$

where $F(x_1, x_2)$ is a formal power series in two variables over the ring E^*, gives the universal formula

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

This formal power series F is an example of a formal group law over the graded ring E^*.

Definition 2.1. A *formal group law* over a ring R is a formal power series $F \in R[\![x_1, x_2]\!]$ satisfying the following conditions:

- $F(x, 0) = F(0, x) = x$ (Identity)
- $F(x_1, x_2) = F(x_2, x_1)$ (Commutativity)
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$ (Associativity)

If R is a graded ring, we require F to be homogeneous of degree 2 where $|x_1| = |x_2| = 2$.

[1] The complex structure induces an orientation on V_x, hence there is a canonical homotopy class of map $\varphi_x : \mathbb{R}^{2n} \to V_x$.

One easily verifies that the power series giving $c_1(L_1 \otimes L_2)$ is a formal group law. The three properties in the definition follow immediately from the natural transformations which give the identity, commutativity, and associativity properties of the tensor product.

Example 2.2. As noted above, the formal group law obtained from ordinary cohomology is $F_+ = F(x_1, x_2) = x_1 + x_2$, and is known as the additive formal group law.

Example 2.3. The multiplicative formal group law is defined by

$$F_\times(x_1, x_2) = x_1 + x_2 - x_1 x_2.$$

One can explicitly verify it satisfies the definition of a formal group law. One can also see that it is obtained from the standard complex orientation of K-theory. Since K-theory is even periodic, we place the classes c_1 in degree 0. The resulting formal group law is over the ring $K^0(\mathrm{pt}) = \mathbb{Z}$ and involves no grading. (Though, we could use the Bott element and its inverse to maintain the grading $|c_1| = 2$ if we wish.)

To the universal line bundle $\xi \to \mathbb{CP}^\infty$, we define the universal first Chern class to be $1 - [\xi] \in K^0(\mathbb{CP}^\infty)$. The term 1 is included so that trivial bundles have trivial first Chern class. Hence, for any line bundle $L \to X$,

$$c_1(L) = 1 - [L] \in K^0(X).$$

A simple calculation demonstrates

$$\begin{aligned} c_1(L_1 \otimes L_2) &= 1 - L_1 \otimes L_2 \\ &= (1 - L_1) + (1 - L_2) - (1 - L_1)(1 - L_2) \\ &= c_1(L_1) + c_1(L_2) - c_1(L_1)c_1(L_2), \end{aligned}$$

demonstrating that the multiplicative formal group law is obtained from K-theory.

Any ring homomorphism $R \to S$ induces a map of formal group laws $FGL(R) \to FGL(S)$. In fact, there is a universal formal group law $F_{univ} \in R_{univ}[\![x_1, x_2]\!]$ such that any $F \in FGL(R)$ is induced by a ring homomorphism $R_{univ} \to R$. The existence of R_{univ} is easy, since one can construct it formally by

$$R_{univ} = \mathbb{Z}[a_{ij}]/\sim$$

where a_{ij} is the coefficient of $x_1^i x_2^j$, and $\sim$ represents all equivalence relations induced by the three axioms of a formal group law. Though this description is quite unwieldy, a theorem by Lazard shows that this ring is isomorphic to a polynomial algebra; i.e.

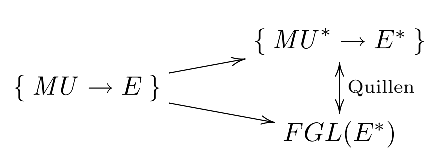
$$R_{univ} \cong L := \mathbb{Z}[a_1, a_2, \ldots]$$

where $|a_i| = -2i$ if we include the grading.

A complex orientation of E therefore induces a map $L \to E^*$ defining the formal group law. Earlier we noted that complex orientations are basically equivalent to maps of ring spectra $MU \to E$, so MU has a canonical complex orientation given by the identity map $MU \to MU$. The following important theorem of Quillen shows that in addition to MU being the universal complex oriented cohomology theory, it is also the home of the universal formal group law. It also explains the grading of the Lazard ring.

Theorem 2.4. *(Quillen) The map $L \to MU^*$ induced from the identity map $MU \to MU$ is an isomorphism.*

To summarize, we have maps



where E^* can be any graded ring. Given a formal group law, can we construct a complex oriented cohomology theory with that formal group law? We will return to this question in Section 4 and see that in certain cases we can construct such a cohomology theory.

First, we discuss formal group laws from the slightly different viewpoint of complex genera. A *genus* is some multiplicative bordism invariant associated to manifolds. There are two main types of genera, and this is due to the description of the cobordism groups from (1).

Definition 2.5. A *complex genus* is a ring homomorphism

$$\varphi : MU^* \to R.$$

An *oriented genus* (or usually just *genus*) is a ring homomorphism

$$\varphi : MSO^* \otimes \mathbb{Q} \to R,$$

where R is a $\mathbb{Q}$-algebra. More explicitly, $\varphi(M)$ only depends on the cobordism class of M and satisfies

$$\varphi(M_1 \sqcup M_2) = \varphi(M_1) + \varphi(M_2), \quad \varphi(M_1 \times M_2) = \varphi(M_1)\varphi(M_2).$$

Quillen's theorem implies there is a 1-1 correspondence between formal group laws over R and complex genera over R. We introduce some common terminology which will make this correspondence more concrete.

First, a homomorphism between formal group laws $F \xrightarrow{f} G$ (over R) is a power series $f(x) \in R[\![x]\!]$ such that

$$f(F(x_1, x_2)) = G(f(x_1), f(x_2)).$$

If f is invertible then it is considered an isomorphism, and f is a strict isomorphism if $f(x) = x +$ higher order terms.

Example 2.6. We could have chosen complex orientation of K-theory so that $c_1(L) = [L] - 1$ as opposed to $1 - [L]$. The resulting formal group law would have been $F(x_1, x_2) = x_1 + x_2 + x_1x_2$, which is also sometimes defined as the multiplicative formal group law. These two formal group laws are (non-strictly) isomorphic, with isomorphism given by $f(x) = -x$. Our original choice, though, coincides with the Todd genus and with conventions in index theory.

Remark 2.7. In general, our formal group law depends on the particular complex orientation. Two different orientations will lead to an isomorphism between the formal group laws. More abstractly, to any complex orientable theory is canonically associated a *formal group*. The choice of orientation gives a coordinate for the formal group, and the formal group expanded in this coordinate is the formal group law.

Over a $\mathbb{Q}$-algebra, any formal group law is uniquely (strictly) isomorphic to the additive formal group law F_+. We denote this isomorphism by $\log_F$ and its inverse by $\exp_F$:

$$F \underset{\exp_F}{\overset{\log_F}{\rightleftarrows}} F_+$$

The isomorphism $\log_F$ can be solved by the following:

$$\begin{aligned} f(F(x_1, x_2)) &= f(x_1) + f(x_2) \\ \left.\frac{\partial}{\partial x_2}\right|_{x,0} (f(F(x_1, x_2))) &= \left.\frac{\partial}{\partial x_2}\right|_{x,0} (f(x_1) + f(x_2)) \\ f'(x)\frac{\partial F}{\partial x_2}(x, 0) &= 1 \\ \log_F(x) = f(x) &= \int_0^x \frac{dt}{\frac{\partial F}{\partial x_2}(t, 0)} \end{aligned} \tag{2}$$

Going from the third to fourth line involves inverting a power series, so one must work over a $\mathbb{Q}$-algebra. If R is torsion-free, then $R \hookrightarrow R \otimes \mathbb{Q}$ is an injection, and we lose no information in considering $\log_F$ instead of F itself.

Over the ring $MU^* \otimes \mathbb{Q}$, the universal formal group law F_{MU} coming from complex cobordism has the particularly nice logarithm

$$\log_{F_{MU}}(x) = \sum_{n \geq 0} \frac{[\mathbb{CP}^n]}{n+1} x^n.$$

Therefore, a formal group law F (or a complex genus) induced by $\varphi : MU^* \to R$ has a logarithm

$$\log_F(x) = \sum_{n \geq 0} \frac{\varphi([\mathbb{CP}^n])}{n+1} x^n.$$

While (modulo torsion in R) the logarithm encodes the value of a genus on any complex manifold, in practice it is difficult to decompose the bordism class of a manifold into projective spaces. However, there is an easier approach to calculating genera due to work of Hirzebruch.

Proposition 2.8. *(Hirzebruch) For R a $\mathbb{Q}$-algebra, there are bijections*

$$\{Q(x) = 1 + a_1 x + a_2 x^2 + \cdots \in R[\![x]\!]\} \longleftrightarrow \{\varphi : MU^* \otimes \mathbb{Q} \to R\}$$
$$\{Q(x) = 1 + a_2 x^2 + a_4 x^4 + \cdots \in R[\![x]\!] \mid a_{odd} = 0\} \longleftrightarrow \{\varphi : MSO^* \otimes \mathbb{Q} \to R\}$$

The first bijection is given by the following construction. Given $Q(x)$, to a complex line bundle $L \to X$ assign the cohomology class

$$\varphi_Q(L) := Q(c_1(L)) \in H^*(X; R).$$

Using the splitting principle, φ_Q extends to a stable exponential characteristic class on all complex vector bundles. The complex genus φ generated by $Q(x)$ is then defined by

$$\varphi(M) := \langle \varphi_Q(TM), [M] \rangle \in R,$$

where M is a stably almost complex manifold, $\langle , \rangle$ is the natural pairing between cohomology and homology, and $[M]$ is the fundamental class (an almost complex structure induces an orientation). Going the other direction, the series $Q(x)$ is related to the formal group law by

$$Q(x) = \frac{x}{\exp_\varphi(x)},$$

where $\exp_\varphi(x)$ is the inverse to $\log_\varphi(x)$. The second bijection follows in the same manner, but one needs an even power series to define the stable exponential characteristic class for real vector bundles.

Example 2.9 (K-theory and $F_\times$). From (2), the logarithm for the multiplicative formal group law $F_\times(x_1, x_2) = x_1 + x_2 - x_1 x_2$ is given by

$$\log_\times(x) = \int_0^x \frac{dt}{1-t} = -\log(1-x).$$

Therefore,

$$\exp_\times(x) = 1 - e^{-x},$$

and the associated power series

$$Q_\times(x) = \frac{x}{\exp_\times(x)} = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \in \mathbb{Q}[\![x]\!]$$

generates the Todd genus Td. When we evaluate the Todd genus on a Riemann surface M^2 with genus g,

$$\begin{aligned} Td(M^2) &= \langle Q(c_1(TM)), [M]\rangle = \langle 1 + \frac{1}{2}c_1(TM) + \cdots, [M]\rangle \\ &= \frac{1}{2}\langle c_1(TM), [M]\rangle = 1 - g. \end{aligned}$$

In this situation, the Todd genus recovers the standard notion of genus.

Note that even though we started with a $\mathbb{Z}$-valued complex genus, the power series $Q(x)$ has fractional coefficients. If one is only given $Q(x)$, it is quite surprising that Todd genus gives integers when evaluated on manifolds with an almost complex structure on the stable tangent bundle. Another explanation for the integrality is given by the following important index theorem. In fact, most of the common genera are equal to the index of some elliptic operator on a manifold (possibly with G-structure).

Theorem 2.10. *(Hirzebruch-Riemann-Roch) Let M be a compact complex manifold, and let V be a holomorphic vector bundle. Then, the index of the Dolbeault operator $\bar{\partial} + \bar{\partial}^*$ on the Dolbeault complex $\{\Lambda^{0,i} \otimes V\}$, which equals the Euler characteristic in sheaf cohomology $H^*(M, V)$, is given by*

$$\text{index}(\bar{\partial} + \bar{\partial}^*) = \chi(M, V) = \langle Td(M)ch(V), [M]\rangle \in \mathbb{Z}.$$

3. Elliptic genera

Another example of a formal group law comes from the group structure of the Jacobi quartic elliptic curve. We first start by working over $\mathbb{C}$. Assume $\delta, \epsilon \in \mathbb{C}$ and the discriminant $\Delta = \epsilon(\delta^2 - \epsilon)^2 \neq 0$. Letting the subscript J stand for Jacobi, we define

$$\log_J(x) := \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}} = \int_0^x \frac{dt}{\sqrt{R(t)}}. \tag{3}$$

Here, $\log_J(x)$ is an example of an elliptic integral, and it naturally arises in physical problems such as modeling the motion of a pendulum. Expanding $\log_J$ as a power series in x produces a formal group law with a nice geometric description. Inverting the function $\log_J(x)$ gives

$$f(z) := \exp_J(z) = (\log_J)^{-1}(z),$$

which is an elliptic function (i.e. periodic with respect to a lattice $\Lambda \subset \mathbb{C}$) satisfying the differential equation $(f')^2(z) = R(z)$. Hence, it parameterizes the elliptic curve C defined by the Jacobi quartic equation

$$y^2 = R(x) = 1 - 2\delta x^2 + \epsilon x^4 \subset \mathbb{CP}^2$$

via the map

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{CP}^2 \\ z &\longmapsto [x(z), y(z), 1] = [f(z), f'(z), 1]. \end{aligned}$$

The additive group structure on the torus $\mathbb{C}/\Lambda$ induces a natural group structure on the elliptic curve C. This group structure coincides with the one given in Chapter 2, defined by $P+Q+R = 0$ for points P, Q, R on a straight line. Near the point $[0, 1, 1]$, the group structure is given in the parameter x by

$$F_J(x_1, x_2) := f(f^{-1}(x_1) + f^{-1}(x_2)) = \exp_J(\log_J(x_1) + \log_J(x_2)).$$

The formal group law F_J defined by the logarithm $\log_J$ can therefore be expressed by

$$\int_0^{x_1} \frac{dt}{\sqrt{R(t)}} + \int_0^{x_2} \frac{dt}{\sqrt{R(t)}} = \int_0^{F_J(x_1,x_2)} \frac{dt}{\sqrt{R(t)}}.$$

Despite the integral $\log_J$ having no closed form solution, the formal group law was solved for explicitly by Euler.

Theorem 3.1. *(Euler)*

$$F_J(x_1,x_2)=\frac{x_1\sqrt{R(x_2)}+x_2\sqrt{R(x_1)}}{1-\epsilon x_1^2x_2^2}.$$

While we previously worked over the field $\mathbb{C}$, the Jacobi quartic is defined over an arbitrary ring, and the universal curve is defined by the same equation over the ring $\mathbb{Z}[\delta,\epsilon]$. The formal group law F_J can be expanded as a power series in the ring $\mathbb{Z}[\frac{1}{2},\delta,\epsilon]$. Any genus whose logarithm is of the form (3) is called an *elliptic genus*, and the universal elliptic genus φ_J corresponds to Euler's formal group law F_J over $\mathbb{Z}[\frac{1}{2},\delta,\epsilon]$. When considering the grading, $|\delta|=-4$ and $|\epsilon|=-8$, so φ_J also defines an oriented genus. In fact, one can calculate that

$$\varphi_J(\mathbb{CP}^2)=\delta,\quad \varphi_J(\mathbb{HP}^2)=\epsilon.$$

Example 3.2. The geometric description of F_J assumed $\Delta=\epsilon(\delta^2-\epsilon)^2\neq 0$ so that the curve C has no singularities. However, the degenerate case $\delta=\epsilon=1$ gives the L-genus, which equals the signature of an oriented manifold:

$$\log(x)=\int_0^x\frac{dt}{1-t^2}=\tanh^{-1}(x),$$
$$Q(x)=\frac{x}{\tanh x}.$$

Similarly, letting $\delta=-\frac{1}{8},\epsilon=0$, we recover the $\widehat{A}$-genus, which for a spin manifold is the index of the Dirac operator:

$$\log(x)=\int_0^x\frac{dt}{\sqrt{1+(t/2)^2}}=2\sinh^{-1}(x/2);$$
$$Q(x)=\frac{x/2}{\sinh(x/2)}.$$

The genera L and $\widehat{A}$ are elliptic genera corresponding to singular elliptic curves. This is explicitly seen in the fact that their logarithms invert to singly-periodic functions as opposed to doubly-periodic functions.

The signature was long known to satisfy a stronger form of multiplicativity, known as strict multiplicativity. If M is a fiber bundle over B with fiber F and connected structure group, then $L(M)=L(B)L(F)$. The same statement holds for the $\widehat{A}$-genus when F is a spin manifold. As more examples were discovered, Ochanine introduced the notion of elliptic genera to explain the phenomenon and classify strictly multiplicative genera.

Theorem 3.3 (Ochanine, Bott–Taubes)**.** *A genus φ satisfies the strict multiplicativity condition $\varphi(M)=\varphi(B)\varphi(F)$ for all bundles of spin manifolds with connected structure group if and only if φ is an elliptic genus.*

There is extra algebraic structure encoded within the values of the universal elliptic genus. Using the Weierstrass $\wp$ function, to a lattice $\Lambda_\tau=\mathbb{Z}\tau+\mathbb{Z}$ we can canonically associate coefficients $\epsilon(\tau)$ and $\delta(\tau)$ satisfying the Jacobi quartic equation. The functions $\epsilon(\tau)$ and $\delta(\tau)$ are modular forms, of weight 2 and 4 respectively, on the subgroup $\Gamma_0(2):=\{A\in SL_2(\mathbb{Z})\,|\,A=\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)\mod 2\}\subset SL_2(\mathbb{Z})$. Therefore, the elliptic genus φ_J associates to any compact oriented $4k$-manifold a modular form of weight $2k$ on the subgroup $\Gamma_0(2)$. Because modular forms are holomorphic and invariant under the translation $\tau\mapsto\tau+1$, we can expand them in the variable $q=e^{2\pi i\tau}$ and consider $\varphi_J(M)\in\mathbb{Q}[\![q]\!]$.

Using insights from quantum field theory, Witten gave an alternate interpretation of the elliptic genus which illuminated several of its properties. His definition of φ_J is as follows. Let M be a spin manifold of dimension n with complex spinor bundle $S(TM^{\mathbb{C}})$. To a complex vector bundle $V\to X$, use the symmetric and exterior powers to define the bundle operations

$$S_tV=1+tV+t^2V^{\otimes 2}+\cdots\in K(X)[\![t]\!],\quad \Lambda_tV=1+tV+t^2\Lambda^2V+\cdots\in K(X)[\![t]\!].$$

Then, the power series in q defined by

$$\Big\langle \widehat{A}(TM)ch\Big(S(TM^{\mathbb{C}})\bigotimes_{l=1}^{\infty} S_{q^l}(TM^{\mathbb{C}}-\mathbb{C}^n)\otimes\bigotimes_{l=1}^{\infty}\Lambda_{q^l}(TM^{\mathbb{C}}-\mathbb{C}^n)\Big),[M^n]\Big\rangle \in \mathbb{Q}[\![q]\!]$$

is equal to the q-expansion of the elliptic genus φ_J. When M is a spin manifold, Witten formally defined the signature operator on the free loop space LM, and he showed its S^1-equivariant index equals φ_J (up to a normalization factor involving the Dedekind η function). The S^1-action on $LM = Map(S^1, M)$ is induced by the natural action of S^1 on itself.

Witten also defined the following genus, now known as the *Witten genus*:

$$\varphi_W(M) := \Big\langle \widehat{A}(TM)ch\Big(\bigotimes_{l=1}^{\infty} S_{q^l}(TM^{\mathbb{C}}-\mathbb{C}^n))\Big),[M^n]\Big\rangle \in \mathbb{Q}[\![q]\!].$$

When M admits a string structure (i.e. M is a spin manifold with spin characteristic class $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$), Witten formally defined the Dirac operator on LM and showed its S^1-equivariant index equals $\varphi_W(M)$, up to a normalization involving η. If M is spin, then the q-series $\varphi_J(M)$ and $\varphi_W(M)$ both have integer coefficients. If M is string, $\varphi_W(M)$ is the q-expansion of a modular form over all of $SL_2(\mathbb{Z})$. Note that the integrality properties can be proven by considering φ_J and φ_W as a power series where each coefficient is the index of a twisted Dirac operator on M. For more detailed information on elliptic genera, an excellent reference is the text [HBJ].

4. Elliptic Cohomology

There is an important description of K-theory via Conner-Floyd. As described in Section 2, the formal group law for K-theory is given by a map $MP^0(\text{pt}) \to K^0(\text{pt}) \cong \mathbb{Z}$ or $MU^* \to K^* \cong \mathbb{Z}[\![\beta,\beta^{-1}]\!]$, depending on our desired grading convention. From this map of coefficients encoding the formal group law, one can in fact recover all of K-theory.

Theorem 4.1. *(Conner-Floyd) For any finite cell complex X,*

$$K^*(X) \cong MP^*(X)\otimes_{MP^0}\mathbb{Z} \cong MU^*(X)\otimes_{MU^*}\mathbb{Z}[\![\beta,\beta^{-1}]\!].$$

In general, Quillen's theorem shows that a formal group law F over a graded ring R is induced by a map $MU^* \to R$. Can we construct a complex-oriented cohomology theory E with formal group law F over $E^* \cong R$? Imitating the Conner-Floyd description of K-theory, we can define

$$E^*(X) := MU^*(X)\otimes_{MU^*} R.$$

While E is a functor satisfying the homotopy, excision, and additivity axioms of a cohomology theory, the "long exact sequence of a pair" will not necessarily be exact. This is due to the fact that exact sequences are not in general exact after tensoring with an arbitrary ring. If R is flat over MU^*, then E will satisfy the long exact sequence of a pair and will be a cohomology theory.

The condition that R is flat over MU^* is very strong and not usually satisfied. However, the Landweber exact functor theorem states that R only needs to satisfy a much weaker set of conditions. This criterion, described in more detail in Chapter 5, states one only needs to check that multiplication by certain elements v_i is injective on certain quotients R/I_i. In the case of the elliptic formal group law, the elements v_1 and v_2 can be given explicitly in terms of ϵ and δ, and the quotients R/I_n are trivial for $n > 2$. Therefore, one can explicitly check Landweber's criterion and conclude the following.

Theorem 4.2. *(Landweber, Ravenel, Stong) There is a homology theory Ell*

$$Ell_*(X) = MU_*(X)\otimes_{MU_*}\mathbb{Z}[\frac{1}{2},\delta,\epsilon,\Delta^{-1}]$$

whose associated cohomology theory is complex oriented with formal group law given by the Euler formal group law. For finite CW complexes X,

$$Ell^*(X) = MU^*(X)\otimes_{MU^*}\mathbb{Z}[\frac{1}{2},\delta,\epsilon,\Delta^{-1}].$$

In Ell^*, $|\delta| = -4, |\epsilon| = -8$.

The theory *Ell* was originally referred to as *elliptic cohomology*, but it is now thought of as a particular elliptic cohomology theory. If we ignore the grading of δ and ϵ, we can form an even periodic theory by $MP(-) \otimes_{MP} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}]$. This motivates the following definiton.

Definition 4.3. An *elliptic cohomology theory* E consists of:

- A multiplicative cohomology theory E which is even periodic,
- An elliptic curve C over a commutative ring R,
- Isomorphisms $E^0(\mathrm{pt}) \cong R$ and an isomorphism of the formal group from E with the formal group associated to C.

The even periodic theory associated to *Ell* is an elliptic cohomology theory related to the Jacobi quartic curve over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon, \Delta^{-1}]$. An obvious question is whether there is a universal elliptic cohomology theory; this universal theory should be related to a universal elliptic curve. Any elliptic curve C over R is isomorphic to a curve given in affine coordinates by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in R.$$

However, there is no canonical way to do this since the Weierstrass equation has non-trivial automorphisms. There is no single universal elliptic curve, but instead a *moduli stack* of elliptic curves, as seen in Chapter 4. Because of this, there is no universal elliptic cohomology theory in the naive sense.

What one does end up with as the "universal elliptic cohomology theory" is *topological modular forms* or *tmf*. Its mere existence is a difficult and subtle theorem, and it will take the rest of these proceedings to construct *tmf*. Roughly speaking, one uses the Landweber exact functor theorem to form a pre-sheaf of elliptic cohomology theories on the moduli stack of elliptic curves. One then lifts this to a sheaf of E_∞ ring-spectra and takes the global sections to obtain the spectrum *tmf*. While constructed out of elliptic cohomology theories, *tmf* is not an elliptic cohomology theory, as evidenced by the following properties.

There is a homomorphism from the coefficients tmf^{-*} to the ring of modular forms MF. While this map is rationally an isomorphism, it is neither injective nor surjective integrally. In particular, tmf^{-*} contains a large number of torsion groups, many of which are in odd degrees. Topological modular forms is therefore not even, and the periodic version *TMF* has period $24^2 = 576$ as opposed to 2 (or as opposed to 24, the period of *Ell*). Furthermore, the theory *tmf* is not complex orientable, but instead has an $MO\langle 8\rangle$ or string orientation denoted σ. At the level of coefficients, the induced map $MString^{-*} \to tmf^{-*}$ gives a refinement of the Witten genus φ_W.

$$\begin{array}{ccc} & & tmf^{-*} \\ & \overset{\sigma}{\nearrow} & \downarrow \\ MString^{-*} & \xrightarrow{\varphi_W} & MF \end{array}$$

While a great deal of information about *tmf* has already been discovered, there are still many things not yet understood. As an example, the index of family of (complex) elliptic operators parameterized by a space X naturally lives in $K(X)$, and topologically this is encoded by the complex orientation of K-theroy. Because of analytic difficulties, there is no good theory of elliptic operators on loop spaces. However, it is believed that families indexes for elliptic operators on loop spaces should naturally live in *tmf* and refine the Witten genus. Making mathematical sense of this would almost certainly require a geometric definition of *tmf*, which still does not yet exist despite efforts including [Seg, ST].

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