Math 309: Section 6 Reviews/Tests

Note that these do not cover the material from Chapters 7, 8.

1. True or False:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V, and the linear map $T : V \to W$ is an isomorphism, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ then is a basis of W.
   (b) If $T : V \to W$ is linear, then $\operatorname{Ker} T$ is a subspace of W.
   (c) If T is not linear, then T is onto.
   (d) Let A be a square matrix. If $A^2 = \mathbf{0}$, then $A = \mathbf{0}$.
   (e) If A is invertible, and $AB = \mathbf{0}$, then $B = \mathbf{0}$.
   (f) Suppose $A \in \mathbb{M}(m, n), B \in \mathbb{M}(n, m)$, and $AB = I_m$. Then $BA = I_n$.

2. Perform the following proofs:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent in V, and $T : V \to W$ is a one-to-one linear map, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.
   (b) Suppose $S : U \to V$ and $T : V \to W$ are linear maps which are both onto. Prove that $T \circ S$ is onto.
   (c) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. Show that $\operatorname{Ker} S \subset \operatorname{Ker}(T \circ S)$.
   (d) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. If $\operatorname{Image}(S) \subset \operatorname{Ker} T$, then $T \circ S = \mathbf{0}$.

3. Find a basis for the kernel and image of T, where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear map whose matrix (relative to the standard basis) is
$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$

4. Let $S : \mathbb{P}_2 \to \mathbb{P}_3$ be the map given by $S(p) = p - 2xp$. Show S is linear. Write the matrix of S relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. Find the kernel and the image of S.

5. Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is linear and satisfies
$$T(1) = 1 + x, \quad T(1 + x) = 1 + x + x^2, \quad T(1 + x + x^2) = 1.$$
   (a) Write the matrix of T relative to the basis $\{1, 1 + x, 1 + x + x^2\}$.
   (b) Calculate the kernel and image of T.
   (c) Find the matrix of T relative to the basis $\{1, x, x^2\}$. (You may leave your answer as a product of matrices and inverses.)
   (d) Where S is the linear map from the previous problem, calculate the matrix of $S \circ T$ relative to the standard bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. (You may leave your answer as a product of matrices and inverses.)
   (e) Is T an isomorphism? If so, find T^{-1}.

6. Show the following are linear maps: (*Insert your own favorite linear map*).

7. Let $B = \{(1, 2, 3), (2, -1, 0), (3, 1, -1)\}$ be a basis for $\mathbb{R}^3$. Find the B-coordinates of $(9, 8, 3)$; i.e. find $[(9, 8, 3)]_B$.

8. Suppose that $T : V \to W$ is a linear map, $\dim V = 5$, and $\dim \operatorname{Ker} T = 2$. Show there exists a 3-dimensional subspace $U \subseteq V$ such that $T(U)$ is a 3-dimensional subspace of W. (Hint: start with a basis for $\operatorname{Ker} T$ and form a basis for V.)

1. Consider the following system of linear equations:

$$\begin{aligned} -3y + z &= 1 \\ x + y - 2z &= 2 \\ x - 2y - z &= 3 \end{aligned}$$

Write the coefficient matrix associated to the linear system. Use Gaussian elimination (and write what elementary row operations you use) to put the matrix into reduced echelon form. Write the solution set to the system of linear equations. Write the solution set as a line or plane in a vector space (in what vector space does the solution set live?).

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a finite subset of the vector space V. Write the definition of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What does it mean for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V? Give the definition of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ being linearly independent. By definition, what does it mean for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be a basis?

3. Is $\{\mathbf{0}\}$ linearly independent? Justify your answer.

4. (a) Suppose that $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent in V, and $\mathbf{x} \in V$. Then, is $\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ linearly independent? Yes, no, maybe?

   (b) Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V. Show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ also spans V.

5. Is the set of polynomials $\{x^2, 4x^2-2, 1\}$ linearly independent in $\mathbb{P}_2$? If not, find a subset of $\{x^2, 4x^2-2, 1\}$ which is linearly independent.

6. Which of the following are subspaces? What properties of a subspace do they satisfy or not satisfy? (Here, $p(a)$ means the polynomial evaluated at $x = a$; e.g. if $p = x^2 - 2$, then $p(1) = 1^2 - 2 = -1$.)

$$\{f \in \mathbb{D}^{(2)}(\mathbb{R}) \mid x^4 f'' - f^2 = 0\} \subset \mathbb{D}^{(2)}(\mathbb{R})$$
$$\{f \in \mathbb{D}^{(2)}(\mathbb{R}) \mid x^4 f'' - e^{-x^2} f = 0\} \subset \mathbb{D}^{(2)}(\mathbb{R})$$
$$\{p \in \mathbb{P}_n \mid p(1) = 0\} \subset \mathbb{P}_n$$

7. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V. Show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for V.

8. Consider the subspaces V_1 and V_2 of $\mathbb{R}^3$ defined by

$$\begin{aligned} V_1 &= \{(x, y, z) \mid -3y + z = 0\} \\ V_2 &= \{(x, y, z) \mid x + y - 2z = 0\} \end{aligned}$$

Find a basis for $V_1 \cap V_2$.

9. Listed here are the 8 axioms of a vector space:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	(addition commutative)
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$	(addition associative)
3. $\exists\, \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	(additive identity)
4. $\forall\, \mathbf{v} \in V$, $\exists\, (-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	(additive inverse)
5. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$	(distributive)
6. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	(distributive)
7. $r(s\mathbf{v}) = (rs)\mathbf{v}$	(scalar associative)
8. $1\mathbf{v} = \mathbf{v}$	(scalar identity)

Using only vector space axioms, show the following properties of vector spaces (justify all your steps):

   (a) $\mathbf{v} + (\mathbf{0} + -\mathbf{v}) = \mathbf{0}$

   (b) If $\mathbf{v} + \mathbf{w} = \mathbf{0}$, then $\mathbf{w} = -\mathbf{v}$.

10. Let $S = \{ax^2 + bx + c \in \mathbb{P}_2 \mid a + b - 2c = 0\} \subset \mathbb{P}_2$.
    (a) Show S is a subspace of $\mathbb{P}_2$.
    (b) Find a basis for S.
    (c) What is the dimension of S?

11. (a) Is $\{(1,2,3), (0,1,7), (-1,4,-8), (3,0,4)\}$ a set of linearly independent vectors in $\mathbb{R}^3$?
    (b) Does $\{(1,1,0), (1,0,1), (0,1,1)\}$ span $\mathbb{R}^3$?

12. (10 points) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be set of vectors in the vector space V.
    (a) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to span V.
    (b) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be linearly independent.
    (c) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be a basis for V.

13. (12 points) Consider the following system of linear equations

$$\left\{ \begin{array}{rrrrcr} x_1 & +3x_2 & -x_3 & +6x_4 & = & 7 \\ 2x_1 & +6x_2 & +x_3 & & = & 8 \\ -2x_1 & -6x_2 & & -4x_4 & = & -10 \end{array} \right.$$

    Solve the above system of equations by using Gaussian elimination, and write the solution set.
    Please indicate the elementary row operations you use, e.g. $2R_2 \to R_2$**.**

14. (10 points) Listed here are the 8 axioms of a vector space:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	(addition commutative)
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$	(addition associative)
3. $\exists\, \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	(additive identity)
4. $\forall\, \mathbf{v} \in V$, $\exists\, (-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	(additive inverse)
5. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$	(distributive)
6. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	(distributive)
7. $r(s\mathbf{v}) = (rs)\mathbf{v}$	(scalar associative)
8. $1\mathbf{v} = \mathbf{v}$	(scalar identity)

    Using only vector space axioms, show the following properties of vector spaces (justify all your steps):
    (a) $(\frac{1}{2}\mathbf{v} + \mathbf{w}) + (\frac{1}{2}\mathbf{v} + \mathbf{w}) = \mathbf{v} + 2\mathbf{w}$.
    (b) If $\mathbf{v} + \mathbf{v} = \mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.

15. (15 points) Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

    (a) Is $\{A_1, A_2, A_3\}$ a linearly independent subset of $\mathbb{M}(2,2)$?
    (b) Is $\{A_1, A_2, A_3\}$ a basis for $\mathbb{M}(2,2)$?

16. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2$ are vectors in a vector space V. Prove:
    (a) (10 points) If $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.
    (b) (10 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3\}$ is linearly independent.

17. (a) (12 points) Let

$$S = \left\{ax^2 + bx + c \in \mathbb{P}_2 \;\middle|\; 2a + b - 4c = 0\right\}.$$

    Show that S is a subspace of $\mathbb{P}_2$.

(b) (8 points) Let

$$T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2,2) \;\middle|\; 2ad - 4bc = 0 \right\}.$$

Is T a subspace of $\mathbb{M}(2,2)$? Justify your answer.

18. (13 points) The set of solutions to the system of equations

$$\begin{cases} x_1 \; -x_2 \qquad\qquad +3x_5 = 0 \\ \qquad\qquad x_3 \qquad -4x_5 = 0 \\ \qquad\qquad\qquad x_4 \; +2x_5 = 0 \end{cases}$$

is a vector space, which we will call V (you do not need to show this). Find a basis for V (be sure to check it is a basis) and calculate the dimension of V.

19. (9 points) State the following definitions:
   (a) A map $T : V \to W$ is linear if:
   (b) If $T : V \to W$ is a linear map, the Kernel of T is:
   (c) A function $f : X \to Y$ is onto if:

20. (15 points) Mark the following statements as True or False. You do not need to show any work.
   (a) If $T : V \to W$ is linear, then $\mathbf{0} \in W$ is always an element of $\operatorname{Im} T$.
   (b) If $A \in \mathbb{M}(2,4)$ and $B \in \mathbb{M}(4,5)$, then $AB \in \mathbb{M}(2,5)$.
   (c) If $A \in \mathbb{M}(4,2)$ and $B \in \mathbb{M}(4,5)$, then $AB \in \mathbb{M}(2,5)$.
   (d) There exists an isomorphism $T : \mathbb{P}_5 \to \mathbb{M}(2,3)$.
   (e) There exists an onto map $T : \mathbb{P}_6 \to \mathbb{M}(3,3)$.

21. (12 points) Are the following maps linear? Either prove or disprove linearity.
   (a) $T : \mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ where $T(f) = x^2 f + \sin x$.
   (b) Let $A \in \mathbb{M}(2,2)$ be a fixed matrix. Define $T : \mathbb{M}(2,2) \to \mathbb{M}(2,2)$ by $T(B) = AB$.

22. (10 points) Let $S : V \to W$ and $T : U \to V$ be linear maps such that $S \circ T = \mathbf{0}$. Prove that $\operatorname{Im} T \subseteq \operatorname{Ker} S$. (Reminder: $S \circ T = \mathbf{0}$ means $(S \circ T)(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in U$.)

23. (8 points) Let $T : V \to W$ be a linear map. Prove that $\operatorname{Ker} T$ is a subspace of V.

   On this question, you may use the following theorem proved in your homework: If $T : V \to W$ is linear, $\operatorname{Ker} T = \mathbf{0}$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of linearly independent vectors in V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W.

24. (8 points) Let $T : V \to W$ be a linear map that is one-to-one. Prove there exists a linear map $S : \operatorname{Im} T \to V$ satisfying $(T \circ S)(\mathbf{w}) = \mathbf{w}$ for any $\mathbf{w} \in \operatorname{Im} T$. (Hint: Start by choosing a basis for V.)

25. (8 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & -2 & 3 & 13 \\ 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Find a basis for $\operatorname{Ker} T$ and $\operatorname{Im} T$.

26. (15 points) Fact which you may use: The following is a basis of $\mathbb{R}^3$:

$$C = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, and that

$$T\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right) = -2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find the matrix of T relative to the above basis C.

(b) What is $\dim \operatorname{Ker} T$, $\dim \operatorname{Im} T$?

(c) Using change of basis matrices and part (a), find the matrix of T relative to the standard basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Write your answer as a product of matrices and their inverses.

(d) Using part (c), find $T\left(\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}\right)$. (You will first need to find any inverse in your answer from (c)).

27. (15 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be given by

$$T(a + bx + cx^2) = (a + 2b + c)1 + (c)x + (2a + 4b + 2c)x^2 + (-a - 2b + 3c)x^3.$$

(a) Find the matrix of T relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$.

(b) Find a basis for $\operatorname{Ker} T$ and $\operatorname{Im} T$.

(c) Let $S : \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map whose matrix, relative to the bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$, is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. Use matrix multiplication to calculate $(S \circ T)(2 + 3x^2)$.

28. Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V.

(a) (4 points) Define $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) (4 points) Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent.

(c) (4 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V, what do you know about the dimension of V?

29. (10 points) Let V, W be vector spaces, and let $T : V \to W$ be a linear map. For any $a \in \mathbb{R}$, define $(aT) : V \to W$ by

$$(aT)(\mathbf{v}) := aT(\mathbf{v}).$$

Show that aT is linear.

30. (a) (6 points) Consider the linear map $T : \mathbb{M}(2, 2) \to \mathbb{P}_2$ given by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + 2b + 4d)1 + (b - 2c + d)x + (a + b)x^2.$$

Find the matrix of T relative to the bases $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $\{1, x, x^2\}$.

(b) (6 points) Show that $\{1 + x^2, 2 + x + x^2, -2x, 4 + x\}$ spans $\mathbb{P}_2$.

(c) (6 points) What is $\dim \operatorname{Ker} T$? Find a basis for $\operatorname{Ker} T$.

31. (a) (10 points) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent in V. Determine whether $\{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, 3\mathbf{v}_2 + \mathbf{v}_3\}$ is linearly independent in V.
    (b) (5 points) Does $\{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, 3\mathbf{v}_2 + \mathbf{v}_3\}$ span V? Justify your answer.

32. (10 points) Let V be vector space with $\dim V = 4$. Show that there exists a linear function $T : \mathbb{R}^3 \to V$ with $\dim \operatorname{Im} T = 2$.

33. (15 points) Diagonalize the matrix

$$A = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

i.e. Find all eigenvalues of A, bases for the eigenspaces, and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (you do not have to check it is diagonal).

34. Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V.
    (a) (4 points) Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V.
    (b) (4 points) Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent.
    (c) (4 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, what do you know about the dimension of V?

35. (10 points) Let V, W be vector spaces, and let T_1, T_2 be linear maps $V \to W$. Define $(T_1 + T_2) : V \to W$ by

$$(T_1 + T_2)(\mathbf{v}) := T_1(\mathbf{v}) + T_2(\mathbf{v}).$$

Show that $T_1 + T_2$ is linear.

36. (a) (6 points) Consider the linear map $T : \mathbb{P}_2 \to \mathbb{M}(2, 2)$ given by

$$T(a + bx + cx^2) = \begin{bmatrix} a + c & 2a + b + c \\ -2b & 4a + b \end{bmatrix}.$$

Find the matrix of T relative to the bases $\{1, x, x^2\}$ and $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

    (b) (6 points) Show that $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is linearly independent in $\mathbb{M}(2, 2)$.
    (c) (6 points) What is $\dim \operatorname{Im} T$? What is $\dim \operatorname{Ker} T$?

37. (a) (10 points) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V. Determine whether $\{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, 3\mathbf{v}_2 + \mathbf{v}_3\}$ spans V.
    (b) (5 points) Is $\{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, 3\mathbf{v}_2 + \mathbf{v}_3\}$ linearly independent? Justify your answer.

38. (10 points) Let V be vector spaces with $\dim V = 5$. Show that there exists a linear function $T : V \to \mathbb{R}^3$ with $\dim \operatorname{Im} T = 2$.

39. (15 points) Diagonalize the matrix

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

i.e. Find all eigenvalues of A, bases for the eigenspaces, and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (you do not have to check it is diagonal).