Notes on Kernel/Image

0.1. **Finding Kernel and Image of a matrix.**
Let $A \in \mathbb{M}(m,n)$ be a matrix. The kernel of A (or nullspace)

$$\operatorname{Ker} A = \{\mathbf{x} \in \mathbb{R}^n \,\big|\, A\mathbf{x} = \mathbf{0}\}$$

is just the solution set to the system of homogeneous equations associated to the matrix A. The Image (or range or column space) is

$$\operatorname{Image} A = \{A\mathbf{x} \,\big|\, \mathbf{x} \in \mathbb{R}^n\} = \{\mathbf{y} \in \mathbb{R}^m \,\big|\, \exists \mathbf{x} \text{ satisfying } \mathbf{y} = A\mathbf{x}\},$$

and it is easy to show this equals the span of the columns of A.

To find a basis for the kernel and image of a matrix A:

(1) Using row reduction, put A in reduced row echelon form (RREF).
(2) Once A is in RREF, write down the set of solutions to the linear system (as you did in Chapter 2). You will get a basis vector for each column with a free variable (however the basis vector is *not* the column vector!).
(3) Image A has a basis given by the columns of A which, after row reduction, contain a pivot.

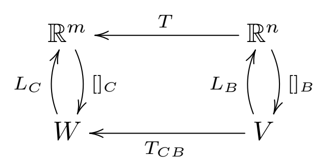
You will always have $\dim \operatorname{Ker} A =$ the number of free variables, and $\dim \operatorname{Image} A =$ number of pivots.

0.2. **Finding Kernel and Image of a linear map.**

$$\operatorname{Ker} T = \{\mathbf{v} \in V \,\big|\, T(\mathbf{v}) = \mathbf{0}\} \subseteq V, \quad \operatorname{Image} T = \{\mathbf{w} \in W \,\big|\, \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\} \subseteq W.$$

(1) If $\mu_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map given by $\mu_A(\mathbf{x}) = A\mathbf{x}$, then $\operatorname{Ker} \mu_A = \operatorname{Ker} A$, and $\operatorname{Image} \mu_A = \operatorname{Image} A$. You are done.
(2) For a general linear map $T : V \to W$, choose a bases B, C of V, W. Determine the matrix T_{CB}, which represents the linear map relative to the bases B and C.
(3) Find a basis for kernel/image of the matrix T_{CB}.
(4) For each vector in the basis of $\operatorname{Ker} T_{CB} \subset \mathbb{R}^n$, map it to V by L_B. Here, $L_B(r_1, \ldots, r_n) = r_1\mathbf{v}_1 + \cdots r_n\mathbf{v}_n$ where $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
(5) For each vector in the basis of $\operatorname{Image} T_{CB} \subset \mathbb{R}^m$, map it to W by L_C.
(6) Note that $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T_{CB} = \#$ free variables, and $\dim \operatorname{Image} T = \dim \operatorname{Image} T_{CB} = \#$ pivots.

Picture:



Remark 1. Equivalently, you can set up the equations

$$T(\mathbf{v}) = \mathbf{0}, \quad T(\mathbf{v}) = \mathbf{w}$$

and solve. Solutions $\mathbf{v}$ to the first equation are elements of $\operatorname{Ker} T$. Vectors $\mathbf{w}$, such that there exists a solution to the second equation, are elements of $\operatorname{Image} T$. In the process of solving, you will find yourself (maybe without realizing it) going through the process given above.

Theorem 1 (Rank-Nullity). *If $T : V \to W$ is linear, and V is finite-dimensional, then*

$$\dim \operatorname{Ker} T + \dim \operatorname{Image} T = \dim V.$$

Corollary 1. *If $T : V \to W$ is linear, and V is finite-dimensional, then*

$$\dim \ker T \geq \dim V - \dim W,$$
$$\dim \operatorname{Image} T \leq \dim V.$$

If T is one-to-one, then $\dim V \leq \dim W$.
If T is onto, then $\dim V \geq \dim W$.