Test 1 Review Solutions

1. $\begin{cases} -3y + z = 1 \\ x + y - 2z = 2 \\ x - 2y - z = 3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 0 & -3 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & -1 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & -2 & -1 & 3 \end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 1 & 1 \end{bmatrix}$

$\xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1/3 R_2 \to R_2} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5/3 & 7/3 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution Set $= \{(7/3 + 5/3\, r,\ -1/3 + 1/3\, r,\ r) \mid r \in \mathbb{R}\} \subset \mathbb{R}^3$

This is line in $\mathbb{R}^3$ $\quad = \{(7/3, -1/3, 0) + r(5/3, 1/3, 1) \mid r \in \mathbb{R}\}$

3. $\{\vec{0}\}$ is linearly dependent.

<u>Proof</u> $\quad 1\vec{0} = \vec{0}$ and $1 \neq 0$, so $\{\vec{0}\}$ not LI.

4. Maybe If $\{\vec{v}, \vec{w}\}$ lin indep, then $\{\vec{v}, \vec{w}, \vec{x}\}$ lin indep iff $\vec{x} \notin \operatorname{span}\{\vec{v}, \vec{w}\}$.

0. One can give examples where $\{\vec{v}, \vec{w}, \vec{x}\}$ is LI and ones where it is Lin Dep.

6. If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans V, then $\{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$ spans V.

<u>Proof</u> Let $\vec{w} \in V$. Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans V, there exist $r_1, \ldots, r_n$

st $\quad \vec{w} = r_1 \vec{v}_1 + \cdots + r_n \vec{v}_n$.

$\Rightarrow \quad \vec{w} = r_1 \vec{v}_1 + \cdots + r_n \vec{v}_n + 0\vec{v}_{n+1}$

$\Rightarrow \quad \vec{w} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$.

$\square$

5. $r_1 \begin{bmatrix} x^2 \\ \\ \end{bmatrix} + r_2 \begin{bmatrix} 4x^2 \\ -2 \end{bmatrix} + r_3 \begin{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} 0x^2 \\ +0x \\ +0 \cdot 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ Free Variable

$\{x^2, 4x^2-2, 1\}$ is Lin Dep, and $1 \in \operatorname{span}\{x^2, 4x^2-2\}$

Above row reduction shows $\{x^2, 4x^2-2\}$ is LI.

6. (a) $S = \{ f \in \mathcal{D}^{(2)}(\mathbb{R}) \mid x^4 f'' - f^2 = 0 \}$ Not a Subspace

$S \neq \emptyset$, since $x^4(0)'' - (0)^2 = 0 - 0 = 0 \Rightarrow 0 \in S$.

Not Closed under Addition Assume $f, g \in S$. Then $(f+g)$ satisfies

$$x^4(f+g)'' - (f+g)^2 = x^4(f''+g'') - (f^2 + 2fg + g^2)$$
$$= (x^4 f'' - f^2) + (x^4 g'' - g^2) + 2fg = 0 + 0 + 2fg = 2fg.$$

$\therefore$ If there is any $f \neq 0$ s.t. $f \in S$, S not closed under addition.

(In fact, $y = 2x^2 \in S$, so S not closed. I won't make you find a non-trivial solution to an ODE on a test though).

Not closed under Scalar Mult Assume $f \in S$, $r \in \mathbb{R}$. Then (rf) satisfies

$$x^4(rf)'' - (rf)^2 = x^4(rf'') - r^2 f^2 = r(x^4 f'' - r f^2),$$ and we don't expect this to be 0 (since it is $x^4 f'' - f^2 = 0$ we know).

(b) $S = \{ f \in \mathcal{D}^{(2)}(\mathbb{R}) \mid x^4 f'' - e^{-x^2} f = 0 \}$ is a subspace

Proof $S \neq \emptyset$ since $x^4(0)'' - e^{-x^2}(0) = 0 \Rightarrow 0 \in S$.

Closed under Add, Scalar Mult Let $f, g \in S$ and $r, s \in \mathbb{R}$. Then $(rf + sg) \in S$ b/c

$$x^4(rf+sg)'' - e^{-x^2}(rf+sg) \overset{\text{Prop of Der}}{=} x^4(rf'' + sg'') - e^{-x^2}(rf+sg)$$
$$= r(x^4 f'' - e^{-x^2} f) + s(x^4 g'' - e^{-x^2} g) \underset{f,g \in S}{=} r(0) + s(0) = 0$$

$\square$

(c) $S = \{ p \in \mathbb{P}_n \mid p(1) = 0 \}$ is a subspace.

Proof $S \neq \emptyset$ Let $p = 0$, Then $0(1) = 0 \Rightarrow 0 \in S$.

Closed under +, Scalar Mult Let $p_1, p_2 \in S$ and $r, s \in \mathbb{R}$. Then

$$(rp_1 + sp_2)(1) \overset{\text{defn}}{=} r(p_1(1)) + s(p_2(1)) \overset{p_1, p_2 \in S}{=} r0 + s0 = 0$$

$\Rightarrow rp_1 + sp_2 \in S$.

$\square$

7. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V, then $\{\vec{v}_1+\vec{v}_2, \vec{v}_1-\vec{v}_2, \vec{v}_3\}$ is a basis for V.

Proof We know $\dim V = 3$, so the Comparison Thm implies $\{\vec{v}_1+\vec{v}_2, \vec{v}_1-\vec{v}_2, \vec{v}_3\}$ is LI iff it spans V. We check LI:

If $r_1(v_1+v_2) + r_2(v_1-v_2) + r_3 v_3 = \bar{0}$, then

$$(r_1+r_2)\vec{v}_1 + (r_1-r_2)\vec{v}_2 + r_3\vec{v}_3 = \bar{0}$$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI, so $\begin{cases} r_1 + r_2 = 0 \\ r_1 - r_2 = 0 \\ r_3 = 0 \end{cases}$ $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Longrightarrow r_1 = r_2 = r_3 = 0.$

Hence $\{\vec{v}_1+\vec{v}_2, \vec{v}_1-\vec{v}_2, \vec{v}_3\}$ is LI and spans $V \Rightarrow$ it is a basis for V. $\square$

8. $V_1 = \{(x,y,z) \mid -3y + z = 0\}$

$V_2 = \{(x,y,z) \mid x+y-2z=0\}$

$\Longrightarrow V_1 \cap V_2$ is solution set of $\begin{cases} -3y + z = 0 \\ x + y - 2z = 0 \end{cases}$

$\begin{bmatrix} 0 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & -1/3 \end{bmatrix}$

$V_1 \cap V_2 = \{(5/3\, r, 1/3\, r, r) \mid r \in \mathbb{R}\} = \text{span}\{(5/3, 1/3, 1)\}$

$(5/3, 1/3, 1) \neq \vec{0}$, so $\{(5/3, 1/3, 1)\}$ is LI $\Rightarrow$ Basis for $V_1 \cap V_2$.

9. a. $\vec{v} + (\vec{0} + -\vec{v}) = \vec{0}$

Proof $\vec{v} + (\vec{0} + -\vec{v}) = (\vec{v} + \vec{0}) + -\vec{v}$ (Add Assoc)

$= \vec{v} + -\vec{v}$ (Add Ident)

$= \vec{0}$ (Add Inverse)

$\square$

b. If $v + \vec{w} = \vec{0}$, then $\vec{w} = -\vec{v}$

Proof. Assume $\vec{v} + \vec{w} = \vec{0}$. Then

$\vec{v} + \vec{w} = \vec{0}$

$-\vec{v} + (\vec{v} + \vec{w}) = -\vec{v} + \vec{0}$ (Adding same thing to both sides)

$(-\vec{v} + \vec{v}) + \vec{w} = -\vec{v}$ (Add Assoc, Add Ident)

$(\vec{v} + -\vec{v}) + \vec{w} = -\vec{v}$ (Add Commut)

$\vec{0} + \vec{w} = -\vec{v}$ (Add Inverse)

$\vec{w} + \vec{0} = -\vec{v}$ (Add Comm)

$\vec{w} = -\vec{v}$ (Add Id)

$\square$

10. (a) $S = \{ax^2 + bx + c \mid a + b - 2c = 0\}$ is a subspace of $\mathbb{P}_2$.

Proof ~~$S \neq \emptyset$~~ $0x^2 + 0x + 0c \in S$ since $0 + 0 - 2 \cdot 0 = 0$.

Closed under +, scalar Let $p_1, p_2 \in S$ where $p_1 = a_1 x^2 + b_1 x + c_1$
$p_2 = a_2 x^2 + b_2 x + c_2$.

Then $(rp_1 + sp_2) = (ra_1 + sa_2)x^2 + (rb_1 + sb_2)x + (rc_1 + sc_2)1 \in S$ b/c

$(ra_1 + sa_2) + (rb_1 + sb_2) - 2(rc_1 + sc_2) =$

$= r(a_1 + b_1 - 2c_1) + s(a_2 + b_2 - 2c_2) = r \cdot 0 + s \cdot 0 = 0.$

$\square$

* Note: This is Conceptually harder than what will be on Test. (5)

10 b. First, let's write solutions to eqn $a+b-2c=0$ $[1\ 1\ -2]$ $r\ s$

Soln Set $= \{(-r+2s, r, s) \mid r,s\in\mathbb{R}\}$

$= \{r(-1,1,0) + s(2,0,1) \mid r,s\in\mathbb{R}\}$

$\therefore S = \{r(-x^2+x) + s(2x^2+1) \mid r,s\in\mathbb{R}\}$

$= \text{span}\{-x^2+x,\ 2x^2+1\}$

These are clearly LI (not a multiple of each other) $\Rightarrow$

$\underline{\{-x^2+x,\ 2x^2+1\}}$ is a basis for S

(c) Basis given by $\{-x^2+x,\ 2x^2+1\}$ $\Rightarrow$ $\dim S = 2$.
2 polys

11. (a) Since $\dim\mathbb{R}^3 = 3$,
Comparison Thm states that (# Lin Indep Vectors) $\leq \dim V$.
Since $\dim\mathbb{R}^3 = 3$ we can't have 4 Lin Indep vectors
$\Rightarrow \{(1,2,3), (0,1,7), (-1,9,-8), (3,0,4)\}$ is Lin Dep.

(b) Does $\{(1,1,0), (1,0,1), (0,1,1)\}$ span $\mathbb{R}^3$? $\underline{\text{Yes}}$

Proof $\dim\mathbb{R}^3 = 3$, so 3 vectors span $\mathbb{R}^3 \Leftrightarrow$ they are LI in $\mathbb{R}^3$.

Check LI:

$$r_1\begin{bmatrix}1\\1\\0\end{bmatrix} + r_2\begin{bmatrix}1\\0\\1\end{bmatrix} + r_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} \rightarrow \begin{bmatrix}1&1&0\\0&-1&1\\0&1&1\end{bmatrix} \rightarrow \begin{bmatrix}1&1&0\\0&1&-1\\0&0&2\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow r_1=r_2=r_3=0$$

$\therefore \{(1,1,0), (1,0,1), (0,1,1)\}$ LI $\Rightarrow$ $\underline{\text{Span } \mathbb{R}^3}$.

$\square$