HOMEWORK DUE MONDAY 4/4

MATH 309, SECTION 6

(1) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given in standard coordinates by the matrix

$$\begin{bmatrix} -1 & 6 \\ \frac{3}{2} & -1 \end{bmatrix}$$

Let $B = \{(1,0),(0,1)\}$ and $B' = \{(-2,1),(2,1)\}$.

(a) Find the change of basis matrices $P_{BB'}$ and $P_{B'B}$ and use these to compute the matrix of T relative to B' (i.e. the above matrix is T_{BB}, and use $P_{BB'}$ and $P_{B'B}$ to find $T_{B'B'}$).

(b) Use $T_{B'B'}$ to find the kernel and image of T.

(2) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be given by

$$T(p) = \begin{bmatrix} p(0) \\ p(2) \end{bmatrix}$$

(e.g. if $p = a + bx$, then $p(4) = a + b(4) = a + 4b$.)

(a) Find the matrix of T relative to the standard bases $B = \{1, x, x^2\}$ of $\mathbb{P}_2$, and $C = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$.

(b) Find the matrix of T relative to the basis $A = \{1, 1+x, 1+x+x^2\}$ of $\mathbb{P}_2$ and $D = \{(1,1),(1,-1)\}$ of $\mathbb{R}^2$.

(3) Suppose that $T : \mathbb{P}_2 \to \mathbb{P}_2$ satisfies

$$T(1+x) = 3(x+x^2), \quad T(x+x^2) = -(x^2+1), \quad T(x^2+1) = 2(x+x^2) + (x+x^2).$$

Calculate $\operatorname{Ker} T$ and $\operatorname{Im} T$.

(Hint: Find the matrix of T relative to the basis $\{1+x, x+x^2, x^2+1\}$, calculate kernel and image relative to that basis, then rewrite the kernel and image as polynomials.)

(4) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T(3,1) = 5(3,1), \quad T(0,2) = -1(0,2).$$

Find the matrix of T relative to the standard basis of $\mathbb{R}^2$.

(5) Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be finite-set of vectors in V, and let $L_B : \mathbb{R}^n \to V$ be the linear map defined by $L(r_1, \ldots, r_n) = \sum_{i=1}^n r_i \mathbf{v}_i$. Prove that L_B is one-to-one if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

(6) 6.7: 1,2

(7) Let V, W be vector spaces with $\dim V = \dim W = 4$. Suppose that $T : V \to W$ is a linear map, and $\dim(\text{Image } T) = 2$. Show there exists a linear map $S : W \to \mathbb{R}^2$ which is onto and which also satisfies $S \circ T = \mathbf{0}$; i.e. $S \circ T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. (Hint: start with a basis for Image T.)