HW DUE 03/28

MATH 309, SECTION 6

Problems: 5.2: 4bc, 6; 6.2: 8ab, 12; 6.4: 2; plus the following:

(1) Show that if $T : V \to W$ is a linear map, then the image of T

$$\text{Im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

is a subspace of W.

(2) Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 & +3x_2 & & +2x_4 \\ & & x_3 & +3x_4 \end{bmatrix}.$$

Calculate and find a basis for both $\operatorname{Ker} T$ and $\operatorname{Im} T$.

(3) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & & +x_3 \\ & x_2 & +x_3 \\ x_1 & +2x_2 & +2x_3 \end{bmatrix}.$$

Find a basis for $\operatorname{Ker} T$ and $\operatorname{Im} T$.

(4) (6.7:4) Suppose $T : V \to W$ is linear and $\operatorname{Ker} T = \{\mathbf{0}\}$. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent subset of V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.

(5) (6.7:5) Suppose $T : V \to W$ is linear and onto. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V. Show that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans W.

Sample problem worked out: Consider the linear operator $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix}.$$

The kernel of T is all (x, y, z) such that $T(x, y, z) = (0, 0, 0)$; i.e. solutions to

$$\begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We easily solve this system using Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column represents a free variable, so we have the solution space is

$$\begin{aligned} \operatorname{Ker} T &= \{(-r, -r, r) \mid r \in \mathbb{R}\} \\ &= \{r(-1, -1, 1) \in \mathbb{R}^3 \mid r \in \mathbb{R}\} \\ &= \mathit{span}\{(-1, -1, 1)\} \end{aligned}$$

Therefore, $(-1, -1, 1)$ is a basis for $\operatorname{Ker} T$, and $(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is an orthonormal basis for $\operatorname{Ker} T$.

To calculate the image, it is easiest to write it as a span of a finite set of vectors. Note that

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

so Image $T = \text{span}\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$. To write a basis for Image T, we need the vectors to be linearly independent. Checking linear independence of $\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$ uses the same Gaussian elimination as above. We quickly see that $\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$ is not linearly independent. The reduced row echelon form shows us $(1, 0, 1)$ and $(0, 1, 1)$ are linearly independent, and that $(1, 1, 2)$ is a linear combination of the other two. Therefore,

$$\mathit{Image}\, T = \mathit{span}\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\} = \mathit{span}\{(1, 0, 1), (0, 1, 1)\},$$

and $\{(1, 0, 1), (0, 1, 1)\}$ are linearly independent, so $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for Image T.