Test 1 Practice Test

Math 309, Section 3

1. Consider the following system of linear equations:

$$\begin{aligned} -3y + z &= 1 \\ x + y - 2z &= 2 \\ x - 2y - z &= 3 \end{aligned}$$

   Write the coefficient matrix associated to the linear system. Use Gaussian elimination (and write what elementary row operations you use) to put the matrix into reduced echelon form. Write the solution set to the system of linear equations.

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a finite subset of the vector space V. Write the definition of span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What does it mean for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V? Give the definition of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ being linearly independent. By definition, what does it mean for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be a basis?

3. Is $\{\mathbf{0}\}$ linearly independent? Justify your answer.

4. (a) Suppose that $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent in V, and $\mathbf{x} \in V$. Then, is $\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ linearly independent? Yes, no, maybe?
   (b) Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V. Show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ also spans V.

5. Is the set of polynomials $\{x^2, 4x^2 - 2, 1\}$ linearly independent in $\mathbb{P}_2$? If not, find a subset of $\{x^2, 4x^2 - 2, 1\}$ which is linearly independent.

6. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V. Show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for V.

7. Suppose V is a subspace of W. Show that if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are linearly independent in V, then they are linearly independent in W.

8. Listed here are the 8 axioms of a vector space:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	(addition commutative)
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$	(addition associative)
3. $\exists\, \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	(additive identity)
4. $\forall\, \mathbf{v} \in V$, $\exists\, (-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	(additive inverse)
5. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$	(distributive)
6. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	(distributive)
7. $r(s\mathbf{v}) = (rs)\mathbf{v}$	(scalar associative)
8. $1\mathbf{v} = \mathbf{v}$	(scalar identity)

   Using only vector space axioms, show the following properties of vector spaces (justify all your steps):
   (a) $\mathbf{v} + (\mathbf{0} + -\mathbf{v}) = \mathbf{0}$
   (b) If $\mathbf{v} + \mathbf{w} = \mathbf{0}$, then $\mathbf{w} = -\mathbf{v}$.

9. Let $S = \{ax^2 + bx + c \in \mathbb{P}_2 \mid a + b - 2c = 0\} \subset \mathbb{P}_2$.
   (a) Show S is a subspace of $\mathbb{P}_2$.
   (b) Find a basis for S.

10. (a) Is $\{(1, 2, 3), (0, 1, 7), (-1, 4, -8), (3, 0, 4)\}$ a set of linearly independent vectors in $\mathbb{R}^3$?
    (b) Does $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ span $\mathbb{R}^3$?

TEST 2 REVIEW

MATH 309, SECTION 3

You should remember the definitions and have a working knowledge of the following concepts already covered: subspaces, linear independence, span, basis, how to solve linear systems, parameterize solution spaces, find a basis for vector spaces/subspaces.

You need to explicitly know: inner products, orthogonality, orthonormality, lengths and angles, orthogonal projection. Linear maps, image, kernel, one-to-one, onto, isomorphism, inverses, composition of maps, matrix of a linear function with respect to a basis.

(1) Let V be an inner product space and $W \subset V$ a finite-dimensional subspace. Show the orthogonal projection $V \to W$ is a linear map.

(2) Let $\langle \cdot, \cdot \rangle$ be the standard inner product (dot product) on $\mathbb{R}^3$. Let $W \subset \mathbb{R}^3$ be the subspace given by the orthonormal basis $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, 1)\}$. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto W.
   (a) Compute $P(a, b, c)$.
   (b) Find the matrix of $P : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the standard basis on $\mathbb{R}^3$.
   (c) Find $\operatorname{Ker} P$.
   (d) Show $P^2 = P$.

(3) Let $\mathbf{v}_1 = (1, 1), \mathbf{v}_2 = (0, 2)$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $\mathbb{R}^2$ produced by applying the Grahm–Schmidt algorithm to $\{\mathbf{v}_1, \mathbf{v}_2\}$.
   (a) Draw $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2$.
   (b) Calculate $\mathbf{e}_1, \mathbf{e}_2$ algebraically using Grahm–Schmidt.

(4) True or False:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are non-zero orthogonal vectors, then they are linearly independent.
   (b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent vectors in V, then they are an orthonormal basis of V.
   (c) If $T : V \to W$ is linear, then $\operatorname{Ker} T$ is a subspace of W.
   (d) If T is not linear, then T is onto.
   (e) Let A be a square matrix. If $A^2 = \mathbf{0}$, then $A = \mathbf{0}$.
   (f) If A is invertible, and $AB = \mathbf{0}$, then $B = \mathbf{0}$.

(5) Perform the following proofs:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent in V, and $T : V \to W$ is a one-to-one linear map, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.
   (b) Suppose $S : U \to V$ and $T : V \to W$ are linear maps which are both onto. Prove that $T \circ S$ is onto.

(c) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. Show that $\operatorname{Ker} S \subset \operatorname{Ker}(T \circ S)$.
(d) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. If $\operatorname{Image}(S) \subset \operatorname{Ker} T$, then $T \circ S = \mathbf{0}$.
(e) Suppose $\langle , \rangle$ is an inner product on the vector space V. Suppose $T : V \to V$ is a linear map such that

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle T(\mathbf{v}_1), T(\mathbf{v}_2) \rangle$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. Prove that $\operatorname{Ker} T = \mathbf{0}$.

(6) Calculate the angle between the functions 1 and x in the inner product space $\mathbb{C}[-1, 1]$.

(7) Find a basis for the kernel and image of T, where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear map whose matrix (relative to the standard basis) is

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$

(8) Let $S : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear maps given by $S(p) = p - 2xp$. Write the matrix of S relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. Find the kernel and the image of T.

(9) Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is linear and satisfies

$$T(1) = 1 + x, \quad T(1 + x) = 1 + x + x^2, \quad T(1 + x + x^2) = 1.$$

(a) Write the matrix of T relative to the basis $\{1, 1 + x, 1 + x + x^2\}$.
(b) Calculate the kernel and image of T.
(c) Calculate $T(a + bx + cx^2)$ and write the matrix of T relative to the basis $\{1, x, x^2\}$.
(d) Using the previous problem, calculate the matrix of $S \circ T$ relative to the bases standard bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$.

(10) Show the following are linear maps: (*Insert your own favorite linear map*).

Test 1

1. (10 points) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be set of vectors in the vector space V.
   (a) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to span V.
   (b) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be linearly independent.
   (c) State the definition of what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be a basis for V.

2. (15 points) Consider the following system of linear equations

$$\begin{cases} x - y + 7z = -1 \\ 2x - y + 11z = -1 \\ -x - y + 3z = -1 \end{cases}$$

   Solve the above system of equations by using Gaussian elimination and then write the solution set. **Please indicate the elementary row operations you use, e.g. $2R_2 \to R_2$.**

3. (15 points) Listed here are the 8 axioms of a vector space:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	(addition commutative)
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$	(addition associative)
3. $\exists\, \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	(additive identity)
4. $\forall\, \mathbf{v} \in V$, $\exists\, (-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	(additive inverse)
5. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$	(distributive)
6. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	(distributive)
7. $r(s\mathbf{v}) = (rs)\mathbf{v}$	(scalar associative)
8. $1\mathbf{v} = \mathbf{v}$	(scalar identity)

   Using only vector space axioms, show the following properties of vector spaces (justify all your steps):
   (a) $\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{v} = \mathbf{v}$.
   (b) If $\mathbf{w} + \mathbf{v} = \mathbf{x}$, then $\mathbf{v} = \mathbf{x} + (-\mathbf{w})$.

4. (15 points) Consider the polynomials $p_1 = x^2 + 2$, $p_2 = x^2 - 1$, $p_3 = x + 1$ in $\mathbb{P}_2$.
   (a) Is $\{p_1, p_2, p_3\}$ a linearly independent subset of $\mathbb{P}_2$?
   (b) Is $\{p_1, p_2, p_3\}$ a basis for $\mathbb{P}_2$?

5. (15 points) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2$ are vectors in the vector space V. Prove:
   If $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subset \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

6. (15 points) Let

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2,2) \;\middle|\; 2a + c - 4d = 0 \right\}$$

   Show that S is a subspace of $\mathbb{M}(2,2)$.

7. (15 points) The set of solutions to the system of equations

$$\left\{ \begin{array}{ccccccc} x_1 & -x_2 & & & +x_5 & = & 0 \\ & & x_3 & & -2x_5 & = & 0 \\ & & & x_4 & +3x_3 & = & 0 \end{array} \right.$$

   is a vector space, which we will call V (you do not need to show this). Find a basis for V (be sure to check it is a basis) and calculate the dimension of V.

Test 2

1. (10 points) Mark the following statements as either True or False. You do not need to show any work.
   (a) If $T : V \to W$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.
   (b) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$, where V is an inner product space. If $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$, then $\mathbf{v}_1 = \mathbf{v}_2$.
   (c) If $A \in \mathbb{M}(2,4)$ and $A' \in \mathbb{M}(2,3)$, then $AA' \in \mathbb{M}(4,3)$.
   (d) if $A \in \mathbb{M}(1,3)$ and $A' \in \mathbb{M}(3,4)$, then $AA' \in \mathbb{M}(1,4)$.

2. (15 points) Consider $\mathbb{R}^3$ with the standard inner product (the dot product). Use Grahm–Schmidt to find an orthonormal basis of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = (1,1,0), \quad \mathbf{v}_2 = (1,3,1).$$

3. (15 points) Let θ denote the angle between the functions $f(x) = 1$ and $g(x) = x$ in the inner product space $\mathbb{C}([0,1])$. Find $\cos\theta$.

4. (a) (10 points) Suppose $S : V \to W$ and $T : U \to V$ are linear maps between vector spaces. Prove that if S and T are both one-to-one, then $S \circ T$ is one-to-one.
   (b) (10 points) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of vectors that spans V, and $T : V \to W$ is a linear map which is onto. Prove that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans W.

5. (10 points) Let $T : \mathbb{D}^{(2)}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ be defined by

$$T(f) = x^2 \frac{d^2 f}{dx^2} + (\cos x) f.$$

Prove that T is linear.

6. (15 points) Consider the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 & -3x_2 & & +4x_4 \\ -x_1 & +3x_2 & +x_3 & +7x_4 \\ & & x_3 & -x_4 \end{bmatrix}.$$

Find a basis for the Kernel of T, and find a basis for the Image of T.

7. (15 points) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be defined by $T(p) = 3p' - p$, where p' is the derivative of p.
Let $S : \mathbb{P}_3 \to \mathbb{P}_3$ be defined by $S(p) = 5p$.
   (a) Find the matrix of T relative to the basis $\{1, x, x^2, x^3\}$.
   (b) Find the matrix of S relative to the basis $\{1, x, x^2, x^3\}$.
   (c) Use parts (a), (b) and matrix multiplication to find the matrix of $S \circ T$ relative to the basis $\{1, x, x^2, x^3\}$.

Test 3a

1. Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V.
   (a) (10 points) Define $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
   Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V.
   (b) (5 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V, what do you know about the dimension of V? (Justify your answer)

2. (10 points) Show that $\{x^2+1, x^2+x+2, -2x\}$ spans $\mathbb{P}_2$.

3. (a) (10 points) Prove: If $\mathbf{v}_1, \mathbf{v}_2 \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subset \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.
   (b) (5 points) Is the converse of (a) true? Either prove it is true or give an example to show it is now true.

4. (10 points) Let $T : V \to W$ be a linear map. Show that $\text{Ker}(T)$ is a subspace of V.

5. (a) (10 points) Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -6 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   Find a basis for $\text{Ker}\, T$ and Image T.
   (b) (5 points) Is T one-to-one? Is T onto? Justify your answer.

6. (10 points) Suppose that $T : V \to W$ is a linear map, $\dim V = 5$, and $\dim \text{Ker}\, T = 2$. Show there exists a 3-dimensional subspace $U \subset V$ such that $T(U)$ is a 3-dimensional subspace of W. (Hint: start with a basis for $\text{Ker}\, T$ and form a basis for V.)

Test 3b

1. Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V.
   (a) (10 points) Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent in V.
   Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V.
   (b) (5 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent in V, what do you know about $\dim V$? Justify your answer.

2. (10 points) Determine whether

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

   spans $\mathbb{M}(2,2)$.

3. (a) (10 points) Prove: If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.
   (b) (5 points) Let $\mathbf{v}_1, \mathbf{v}_2 \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Then is $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent, linearly dependent, or there is not enough information. Justify your answer.

4. (10 points) Let $T : V \to W$ be a linear map. Show that Image(T) is a subspace of W.

5. (10 points) Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -6 \\ -2 & 1 & -3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   Find a basis for $\operatorname{Ker} T$ and Image T.

6. (5 points) Let (the non-linear function) $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x-1)(x-2)(x-3)$. Is f one-to-one? Is f onto? Justify your answer.

7. (10 points) Let V, W be vector spaces with $\dim V = \dim W = 4$. Suppose that $T : V \to W$ is a linear map, and dim(Image T) $= 2$. Show there exists a linear map $S : W \to \mathbb{R}^2$ which is onto and which also satisfies $S \circ T = \mathbf{0}$; i.e. $S \circ T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. (Hint: start with a basis for Image T.)