TEST 2 REVIEW

MATH 309, SECTION 3

You should remember the definitions and have a working knowledge of the following concepts already covered: subspaces, linear independence, span, basis, how to solve linear systems, parameterize solution spaces, find a basis for vector spaces/subspaces.

You need to explicitly know: inner products, orthogonality, orthonormality, lengths and angles, orthogonal projection. Linear maps, image, kernel, one-to-one, onto, isomorphism, inverses, composition of maps, matrix of a linear function with respect to a basis.

(1) Let V be an inner product space and $W \subset V$ a finite-dimensional subspace. Show the orthogonal projection $V \to W$ is a linear map.

(2) Let $\langle \cdot, \cdot \rangle$ be the standard inner product (dot product) on $\mathbb{R}^3$. Let $W \subset \mathbb{R}^3$ be the subspace given by the orthonormal basis $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, 1)\}$. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto W.
   - (a) Compute $P(a, b, c)$.
   - (b) Find the matrix of $P : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the standard basis on $\mathbb{R}^3$.
   - (c) Find $\operatorname{Ker} P$.
   - (d) Show $P^2 = P$.

(3) Let $\mathbf{v}_1 = (1, 1), \mathbf{v}_2 = (0, 2)$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $\mathbb{R}^2$ produced by applying the Grahm–Schmidt algorithm to $\{\mathbf{v}_1, \mathbf{v}_2\}$.
   - (a) Draw $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2$.
   - (b) Calculate $\mathbf{e}_1, \mathbf{e}_2$ algebraically using Grahm–Schmidt.

(4) True or False:
   - (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are non-zero orthogonal vectors, then they are linearly independent.
   - (b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent vectors in V, then they are an orthonormal basis of V.
   - (c) If $T : V \to W$ is linear, then $\operatorname{Ker} T$ is a subspace of W.
   - (d) If T is not linear, then T is onto.
   - (e) Let A be a square matrix. If $A^2 = \mathbf{0}$, then $A = \mathbf{0}$.
   - (f) If A is invertible, and $AB = \mathbf{0}$, then $B = \mathbf{0}$.

(5) Perform the following proofs:
   - (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent in V, and $T : V \to W$ is a one-to-one linear map, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.
   - (b) Suppose $S : U \to V$ and $T : V \to W$ are linear maps which are both onto. Prove that $T \circ S$ is onto.

(c) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. Show that $\operatorname{Ker} S \subset \operatorname{Ker}(T \circ S)$.
(d) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. If $\operatorname{Image}(S) \subset \operatorname{Ker} T$, then $T \circ S = \mathbf{0}$.
(e) Suppose $\langle , \rangle$ is an inner product on the vector space V. Suppose $T : V \to V$ is a linear map such that

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle T(\mathbf{v}_1), T(\mathbf{v}_2) \rangle$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. Prove that $\operatorname{Ker} T = \mathbf{0}$.

(6) Calculate the angle between the functions 1 and x in the inner product space $\mathbb{C}[-1, 1]$.
(7) Find a basis for the kernel and image of T, where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear map whose matrix (relative to the standard basis) is

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$

(8) Let $S : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear maps given by $S(p) = p - 2xp$. Write the matrix of S relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. Find the kernel and the image of T.
(9) Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is linear and satisfies

$$T(1) = 1 + x, \quad T(1 + x) = 1 + x + x^2, \quad T(1 + x + x^2) = 1.$$

(a) Write the matrix of T relative to the basis $\{1, 1 + x, 1 + x + x^2\}$.
(b) Calculate the kernel and image of T.
(c) Calculate $T(a + bx + cx^2)$ and write the matrix of T relative to the basis $\{1, x, x^2\}$.
(d) Using the previous problem, calculate the matrix of $S \circ T$ relative to the bases standard bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$.

(10) Show the following are linear maps: (*Insert your own favorite linear map*).