HW ASSIGNED 10/29 AND 11/1
DUE FRIDAY 11/5

MATH 309, SECTION 3

(1) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by
$$T(x, y, z) = (x + 3z, y + 4z).$$
Calculate $\operatorname{Ker} T$ and find an orthonormal basis for $\operatorname{Ker} T$.

(2) Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by
$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 & +3x_2 & & +2x_4 \\ & & x_3 & +3x_4 \end{bmatrix}.$$
Calculate $\operatorname{Ker} T$ and find an ~~orthonormal~~ basis for $\operatorname{Ker} T$.

(3) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & & +x_3 \\ & x_2 & +x_3 \\ x_1 & +2x_2 & +2x_3 \end{bmatrix}.$$
Calculate $\operatorname{Image} T$ and find a basis for Image T (it doesn't need to be orthonormal).

(4) Consider the linear operator $T : \mathbb{D}^{(2)}(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ given by
$$T(y) = \frac{d^2y}{dx^2} + k^2 y,$$
where $y = y(x) \in \mathbb{D}^{(2)}(\mathbb{R})$. One often uses the notation $T = \frac{d^2}{dx^2} + k^2$. Below, let n be a constant.
   (a) Show that T is linear.
   (b) Compute $T(x^n)$.
   (c) Compute $T(\cos(nx))$ and $T(\sin(nx))$.
   (d) Use part (c) to obtain a 2-dimensional subspace in $\operatorname{Ker} T$. You may use (without proving) the fact that $\cos(nx)$ and $\sin(nx)$ are linearly independent.
   (In fact, this 2-dimensional subspace equals $\operatorname{Ker} T$, but you will have to take Differential Equations to find out why.)

(5) (6.7:4) Suppose $T : V \to W$ is linear and $\operatorname{Ker} T = \{\mathbf{0}\}$. Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent subset of V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.

(6) (6.7:5) Suppose $T : V \to W$ is linear and onto. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V. Show that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ spans W.

(7) (6.7:6) Suppose $T : V \to W$ is linear. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors in V such that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent subset of V.

Sample problem worked out: Consider the linear operator $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix}.$$

The kernel of T is all (x, y, z) such that $T(x, y, z) = (0, 0, 0)$; i.e. solutions to

$$\begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We easily solve this system using Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column represents a free variable, so we have the solution space is

$$\begin{aligned} \operatorname{Ker} T &= \{(-r, -r, r) \mid r \in \mathbb{R}\} \\ &= \{r(-1, -1, 1) \in \mathbb{R}^3 \mid r \in \mathbb{R}\} \\ &= span\{(-1, -1, 1)\} \end{aligned}$$

Therefore, $(-1, -1, 1)$ is a basis for $\operatorname{Ker} T$, and $(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is an orthonormal basis for $\operatorname{Ker} T$.

To calculate the image, it is easiest to write it as a span of a finite set of vectors. Note that

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & & +z \\ & y & +z \\ x & +y & +2z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

so Image $T = \text{span}\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$. To write a basis for Image T, we need the vectors to be linearly independent. Checking linear independence of $\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$ uses the same Gaussian elimination as above. We quickly see that $\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\}$ is not linearly independent. The reduced row echelon form shows us $(1, 0, 1)$ and $(0, 1, 1)$ are linearly independent, and that $(1, 1, 2)$ is a linear combination of the other two. Therefore,

$$Image\, T = span\{(1, 0, 1), (0, 1, 1), (1, 1, 2)\} = span\{(1, 0, 1), (0, 1, 1)\},$$

and $\{(1, 0, 1), (0, 1, 1)\}$ are linearly independent, so $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for Image T.

We can use Grahm–Schmidt to give an orthonormal basis if we wish. Applying Grahm–Schmidt to $\{(1, 0, 1), (0, 1, 1)\}$ gives the orthonormal basis

$$\{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{6}}, \frac{4}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}.$$