Name: ANSWER KEY Section: ____________

Math 234: Sections 1-3

Test 4

April 26, 2010

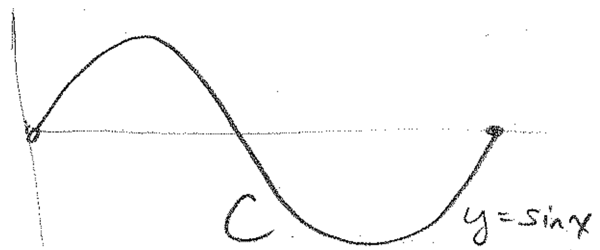
The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

Question	Points	Score
1	13	
2	15	
3	17	
4	15	
5	25	
6	15	
Total:	100	

1. (13 points) A wire with variable density sits inside the xy-plane. The wire is given by the curve $y = \sin x$, $0 \leq x \leq 2\pi$; the density is given by $\delta(x, y) = x + y^2$. Set up integrals (*do not evaluate*) to find

(a) the length of the wire,

(b) the mass of the wire.



$\vec{r}(t) = \langle t, \sin t \rangle, \ 0 \leq t \leq 2\pi$

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 1, \cos t \rangle$

$|\vec{v}| = \sqrt{1 + \cos^2 t}$

Length $= \int_C ds = \int_0^{2\pi} |\vec{v}| \, dt$

$$\boxed{L = \int_0^{2\pi} \sqrt{1 + \cos^2 t} \, dt}$$

MASS $= \int_C \delta \, ds = \int_0^{2\pi} (x + y^2) |\vec{v}| \, dt$

$$\boxed{m = \int_0^{2\pi} (t + \sin^2 t) \sqrt{1 + \cos^2 t} \, dt}$$

$\vec{r_0}$ $\vec{r_1}$

2. (15 points) Let C be the straight-line segment going from $(0, 0, 1)$ to $(3, 1, 0)$. Calculate

$$\int_C (y^2 + 2)\, dx + x\, dy + (z - 1)\, dz$$

<u>Param of C</u>

$$\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) \quad , 0 \le t \le 1$$

$$= \langle 0, 0, 1 \rangle + t\langle 3, 1, -1 \rangle$$

$$= \langle 3t, t, 1-t \rangle$$

$$= \langle x, y, z \rangle$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 3, 1, -1 \rangle$$

$$= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

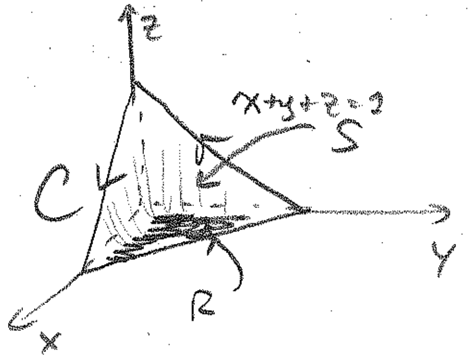
$$\int_C (y^2 + 2)\, dx + x\, dy + (z-1)\, dx$$

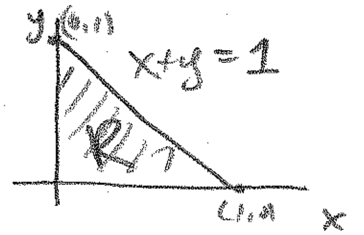
$$= \int_0^1 \left[(t^2+2)3 + 3t + (1-t-1)(-1) \right] dt$$

$$= \int_0^1 3t^2 + 6 + 3t + t \quad dt$$

$$= t^3 + 2t^2 + 6 \Big|_0^1 = \boxed{9}$$

3. (17 points) Let C be the curve given by the intersection of the plane $x+y+z=1$ with the first octant (oriented counter-clockwise when viewed from above). If $\mathbf{F} = 2xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$, use Stokes' theorem to calculate the work done by $\mathbf{F}$ along the curve C.





$$\text{Curl } F = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & xy & yz \end{vmatrix} = \langle z, 2x, y \rangle$$

Surface S is $g(x,y,z) = x+y+z = 1$

$$\nabla g = \langle 1,1,1 \rangle$$

$$|\nabla g \cdot \vec{k}| = |1| = 1$$

Note: We use $+\nabla g$, since curve goes counter-clockwise around this.

$$\int_C \vec{F} \cdot d\vec{r} \stackrel{\text{Stokes}}{=} \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

$$= \iint_R \frac{(\nabla \times \vec{F}) \cdot \nabla g}{|\nabla g \cdot \vec{k}|} \, dA = \iint_R \frac{\langle z, 2x, y \rangle \cdot \langle 1,1,1 \rangle}{1} \, dA$$

$$= \iint_R z + 2x + y \, dA = \int_0^1 \int_0^{1-x} (1-x-y) + 2x + y \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} 1 + x \, dy \, dx = \int_0^1 (1-x^2) \, dx = 1 - 1/3$$

$$= \boxed{2/3}$$

4. (15 points) Consider the conservative force $\mathbf{F} = \langle yz + 2x, xz + 2y, xy + 2z \rangle$.

(a) Find a potential function for $\mathbf{F}$.

(b) Find $\int_C \mathbf{F} \cdot dr$ where C is a curve starting at $(0,0,0)$ and ending at $(1,1,1)$.

(a) $$\frac{\partial f}{\partial x} = yz + 2x$$

$$\Rightarrow f = \int \frac{\partial f}{\partial x}\,dx = \int (yz + 2x)\,dx = xyz + x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz + 2y$$

$$\frac{\partial g}{\partial y} = 2y$$

$$g = y^2 + h(z) \quad \Rightarrow f = xyz + x^2 + y^2 + h$$

$$\frac{\partial f}{\partial z} = xy + \frac{\partial h}{\partial z} = xy + 2z$$

$$\frac{\partial h}{\partial z} = 2z$$

$$h = z^2$$

$$\Rightarrow \boxed{f = xyz + x^2 + y^2 + z^2}$$

(b) $$\int_C \vec{F} \cdot d\vec{r} = \int_{(0,0,0)}^{(1,1,1)} \nabla(xyz + x^2 + y^2 + z^2)\,dt$$

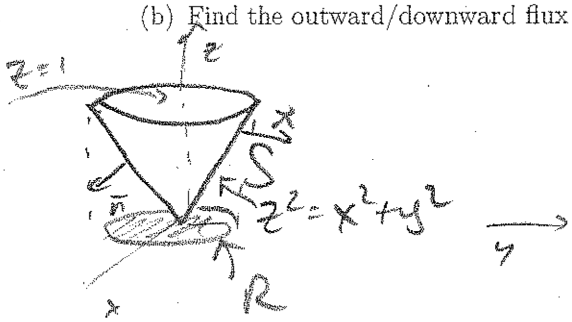
$$= xyz + x^2 + y^2 + z^2 \Big|_{(0,0,0)}^{(1,1,1)} = (1\cdot1\cdot1 + 1^2 + 1^2 + 1^2) - (0+0+0+0)$$

$$= \boxed{4}$$

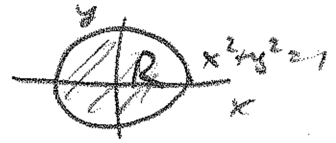
5. (25 points) Let S be the cone $z^2 = x^2 + y^2$, $\quad 0 \leq z \leq 1$. (The cone does not have a top.)

   (a) Find the surface area of the cone S.

   (b) Find the outward/downward flux of $\mathbf{F} = \langle x, y, 0 \rangle$ over S.



R, shadow of $z^2 = x^2+y^2$



Surface S is given by $g = x^2+y^2-z^2 = 0$

$$\nabla g = \langle 2x, 2y, -2z \rangle = 2\langle x, y, -z \rangle$$

$$|\nabla g| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2(x^2+y^2)}$$

$$|\nabla g \cdot \vec{k}| = |-2z| = 2z$$

Note, $+\nabla g$ points out and down.

(a) Surface Area $= \iint_S d\sigma = \iint_R \frac{|\nabla g|}{|\nabla g \cdot \vec{k}|}\, dA$

$$= \iint_R \frac{\cancel{2}\sqrt{2(x^2+y^2)}}{\cancel{2}z}\, dA$$

Note: In polar coords $z^2 = x^2+y^2 = r^2$, $z = r$

$$\overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}\, \cancel{r}}{\cancel{r}}\, r\, dr\, d\theta = \int_0^{2\pi} \sqrt{2}/2\; r^2\Big|_0^1 d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\, d\theta$$

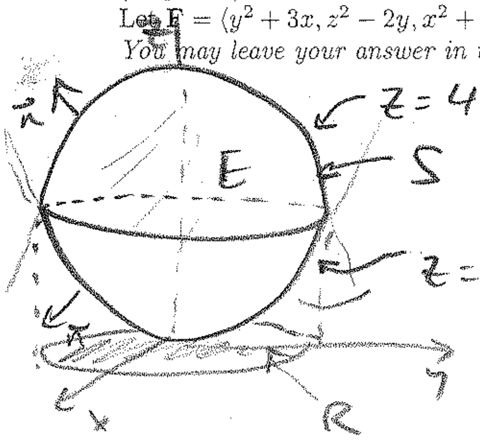
$$= \boxed{\pi\sqrt{2}}$$

(b) Flux $= \iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_R \frac{\vec{F}\cdot \pm\nabla g}{|\nabla g \cdot \vec{k}|}\, dA$

$$= \iint_R \frac{\langle x, y, 0\rangle \cdot 2\langle x, y, -z\rangle}{2z}\, dA = \iint_R \frac{x^2+y^2}{z}\, dA$$

$$\overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^1 \frac{r^2}{\cancel{r}}\, \cancel{r}\, dr\, d\theta = \boxed{\frac{2\pi}{3}}$$

6. (15 points) Let S be the surface whose top half is $z = 4 - 3x^2 - 3y^2$ and whose bottom-half is $z = x^2 + y^2$. Let $\mathbf{F} = \langle y^2 + 3x, z^2 - 2y, x^2 + 2z \rangle$. Use the Divergence Theorem to find the outward flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$. *You may leave your answer in the form of an iterated integral.*

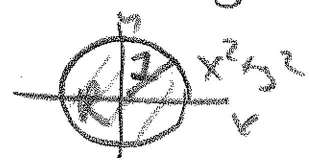


Region R is shadow of

$4 - 3x^2 - 3y^2 = x^2 + y^2$

$4 = 4x^2 + 4y^2$

$1 = x^2 + y^2$



$$\text{Div} \vec{F} = \nabla \cdot F = \nabla \cdot \langle y^2 + 3x, z^2 - 2y, x^2 + 2z \rangle$$
$$= 3 - 2 + 2 = \underline{3}$$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma \overset{\text{Div Thm}}{=\!=} \iiint_E \text{Div} F\, dV$$

$$= \iiint_E 3\, dV = \iint_R \int_{z = x^2+y^2}^{z = 4-3x^2-3y^2} 3\, dV$$

Polar / Cyl

$$= \boxed{\int_0^{2\pi} \int_0^1 \int_{r^2}^{4-3r^2} 3r\, dz\, dr\, d\theta}$$