Name: ANSWER KEY Section: ____________

Math 234: Sections 1-3

Test 3

March 30, 2010

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

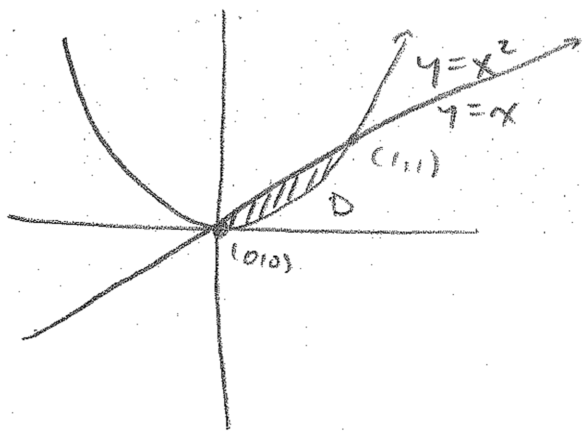
Question	Points	Score
1	15	
2	15	
3	20	
4	15	
5	20	
6	15	
Total:	100	

1. (15 points) Calculate $\iint_D f\,dA$ where $f(x, y) = 2xy$ and D is the region bounded by the curves $y = x^2$ and $y = x$.

Intersection $x^2 = y = x$

$x^2 - x = x(x-1) = 0$

$x = 0, 1$



$$\iint_D f\,dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} 2xy\,dy\,dx$$

$$= \int_0^1 xy^2 \Big|_{y=x^2}^{y=x} dx = \int_0^1 (x^3 - x^5)\,dx$$

$$= \frac{1}{4}x^4 - \frac{1}{6}x^6 \Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{6-4}{24} = \boxed{\frac{1}{12}}$$

2. (15 points) (a) Using *spherical coordinates*, set up an integral to calculate the volume of the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 9$. *Do not evaluate the integral.*

(b) The same as part (a), but this time bounded above by the sphere $x^2+y^2+(z-1)^2 = 1$. (In other words, use spherical coordinates to set up an integral calculating the volume of the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+(z-1)^2 = 1$. *Do not evaluate the integral.*)

Translate Eqns to spher. Coords

$$z = \sqrt{x^2+y^2}$$
$$z^2 = x^2+y^2$$

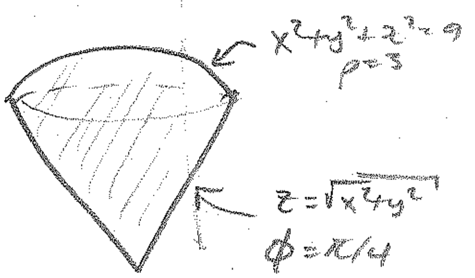
spherical →

$$\rho^2\cos^2\phi = \rho^2\sin^2\phi$$
$$\tan^2\phi = 1$$
$$\underline{\phi = \pi/4}$$

$$x^2+y^2+z^2 = 9$$
$$\rho^2 = 9$$
$$\underline{\rho = 3}$$

$$x^2+y^2+(z-1)^2 = 1$$
$$x^2+y^2+z^2 - 2z + 1 = 1$$
$$x^2+y^2+z^2 = 2z$$
$$\rho^2 = 2\rho\cos\phi$$
$$\underline{\rho = 2\cos\phi}$$

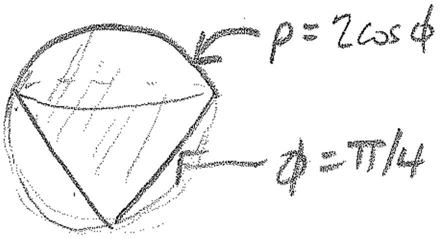
(a)



$$V = \iiint_D dV$$

$$\boxed{= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

(b)



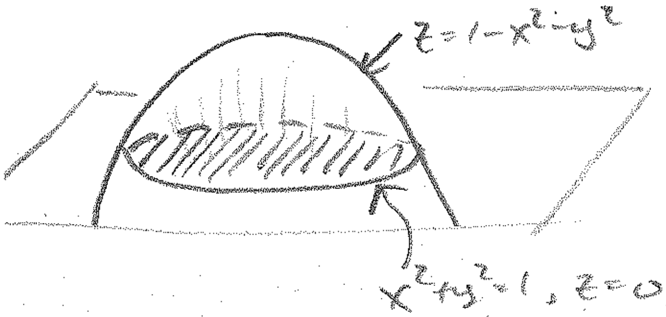
$$V = \iiint_D dV$$

$$= \boxed{\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

3. (20 points) Find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Intersection. $1 - x^2 - y^2 = z = 0$

$x^2 + y^2 = 1$

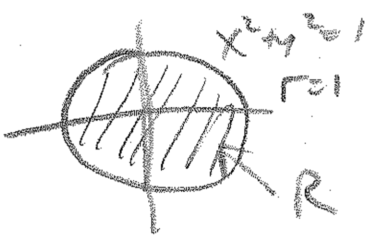


Let R denote $x^2 + y^2 \le 1$ (ie shaded region)

$$V = \iint_R (1 - x^2 - y^2)\, dA$$

$$\text{or} = \iint_R \left[\int_{z=0}^{z=1-x^2-y^2} dz \right] dA = \iint_R (1 - x^2 - y^2)\, dA$$

Use polar coords



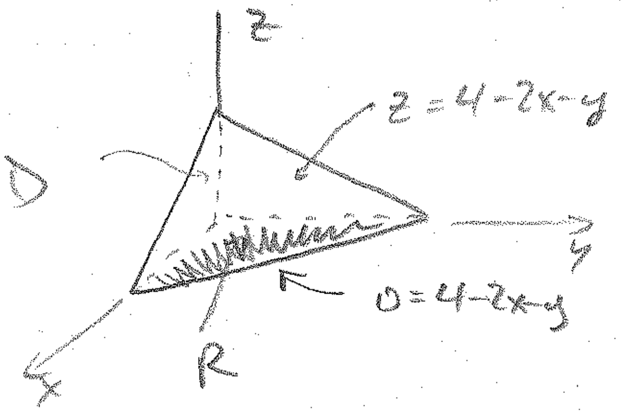
$$\iint_R (1 - x^2 - y^2)\, dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 \right) d\theta$$

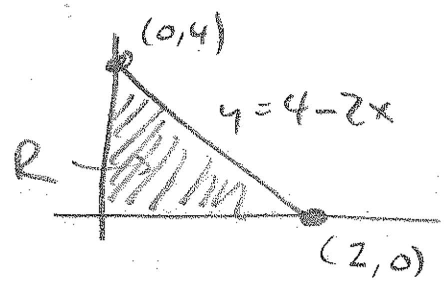
$$= \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

4. (15 points) Let $f(x, y, z) = z \sin x$. Set up an integral to find the average value of f on the region in the first octant (i.e. $x, y, z \geq 0$) bounded by the plane $z = 4 - 2x - y$. *Do not evaluate.*

$$f_{Avg} = \frac{\iiint_D f \, dV}{\iiint_D dV}$$



Project to x-y plane (i.e. z=0)



$$\iiint_D dV = \iint_R \left[\int_{z=0}^{z=4-2x-y} dz \right] dA$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} \int_{z=0}^{z=4-2x-y} dz \, dy \, dx$$

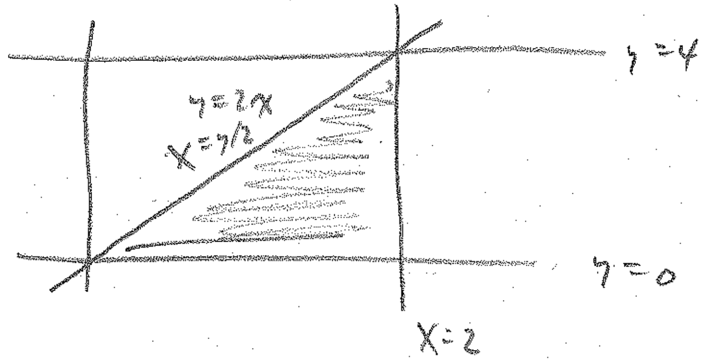
$$\boxed{f_{Avg} = \frac{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \sin x \, dz \, dy \, dx}{\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz \, dy \, dx}}$$

(There are other possible orders of integration)

5. (20 points) Solve the following integral by drawing the bounds of integration, reversing the order of integration, and then finally integrating:

$$\int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy$$

$$\int_{y=0}^{y=4} \int_{x=y/2}^{x=2} e^{x^2}\, dx\, dy = ?$$

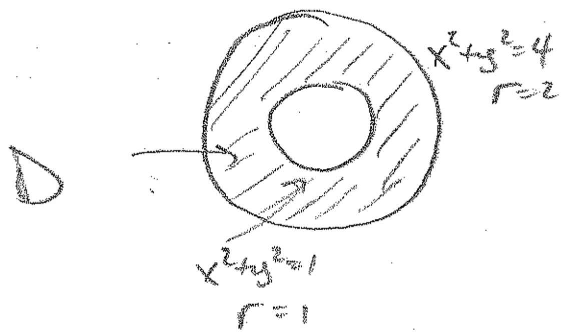


$$= \int_{x=0}^{x=2} \int_{y=0}^{y=2x} e^{x^2}\, dy\, dx$$

$$= \int_0^2 e^{x^2} y \Big|_{y=0}^{y=2x} dx = \int_0^2 e^{x^2} 2x\, dx$$

$$= e^{x^2} \Big|_0^2 = e^4 - e^0 = \boxed{e^4 - 1}$$

6. (15 points) Let D be the region in the xy-plane bounded between the two curves $x^2+y^2=1$ and $x^2+y^2=4$. Suppose the density of D is given by the function $\delta(x,y)=2x^2+y^2$. Using *polar coordinates*, set up integrals to find the center of mass of D. *Do not evaluate.*



Convert eqns

$$x^2+y^2=1 \Rightarrow r^2=1,\ \underline{r=1}$$

$$x^2+y^2=4 \Rightarrow r^2=4,\ \underline{r=2}$$

$$\delta = 2x^2+y^2 = (x^2+y^2)+x^2$$

$$= 2r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(1+\cos^2\theta)$$

$$\bar{x} = \frac{\iint_D x\delta\, dA}{\iint_D \delta\, dA} = \frac{\int_0^{2\pi}\int_1^2 (r\cos\theta)(r^2(1+\cos^2\theta))r\, dr\, d\theta}{\int_0^{2\pi}\int_1^2 r^3(1+\cos^2\theta)\, dr\, d\theta}$$

$$\bar{y} = \frac{\iint_D y\delta\, dA}{\iint_D \delta\, dA} = \frac{\int_0^{2\pi}\int_1^2 (r\sin\theta)(r^2(1+\cos^2\theta))r\, dr\, d\theta}{\int_0^{2\pi}\int_1^2 r^3(1+\cos^2\theta)\, dr\, d\theta}$$