Name: ANSWER KEY Section: ____________

Math 234: Sections 1-3

Test 2

March 2, 2010

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

Question	Points	Score
1	10	
2	12	
3	10	
4	15	
5	10	
6	10	
7	18	
8	15	
Total:	100	

1. (10 points) Suppose a particle moves in the xy-plane with velocity and initial position given by

$$\mathbf{v}(t) = \langle e^{-t}, e^{-2t} \rangle, \quad \mathbf{r}(0) = \langle -3, 5 \rangle.$$

Find the position function $\mathbf{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \int e^{-t}\,dt,\ \int e^{-2t}\,dt \right\rangle$$

$$= \left\langle -e^{-t} + C_1,\ -\tfrac{1}{2}e^{-2t} + C_2 \right\rangle$$

$$\langle -3, 5 \rangle = \vec{r}(0) = \langle -1 + C_1,\ -1/2 + C_2 \rangle$$

$$C_1 = -2$$

$$C_2 = 5 + 1/2 = 11/2$$

$$\boxed{\vec{r}(t) = \left\langle -e^{-t} - 2,\ -\tfrac{1}{2}e^{-2t} + 11/2 \right\rangle}$$

2. Suppose a particle moves in three-dimensional space with position given by

$$\mathbf{r}(t) = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t \rangle, \qquad 2 \le t \le 4.$$

(a) (5 points) Find the velocity and acceleration at time t.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle t, 2t^{1/2}, 2 \rangle$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \langle 1, t^{-1/2}, 0 \rangle$$

(b) (7 points) Find the total distance traveled by the particle between times $t = 2$ and $t = 4$.

$$L = \int_{t=2}^{t=4} |\vec{v}(t)|\, dt = \int_2^4 \sqrt{t^2 + 4t + 4}\, dt$$

$$= \int_2^4 \sqrt{(t+2)^2}\, dt = \int_2^4 t + 2 \quad dt$$

$$= \frac{1}{2}t^2 + 2t \Big|_2^4 = \frac{1}{2}16 + 2 \cdot 4 - \left(\frac{1}{2}4 + 2 \cdot 2\right)$$

$$= 8 + 8 - (2+4) = \boxed{10}$$

3. (10 points) Let $f(x,y) = \dfrac{1}{x^2+y^2}$. Find the domain of f and draw level curves for $f = .25, 1, 4$.

Domain

$$x^2 + y^2 \neq 0$$

$$(x,y) \neq (0,0)$$

Level Curves

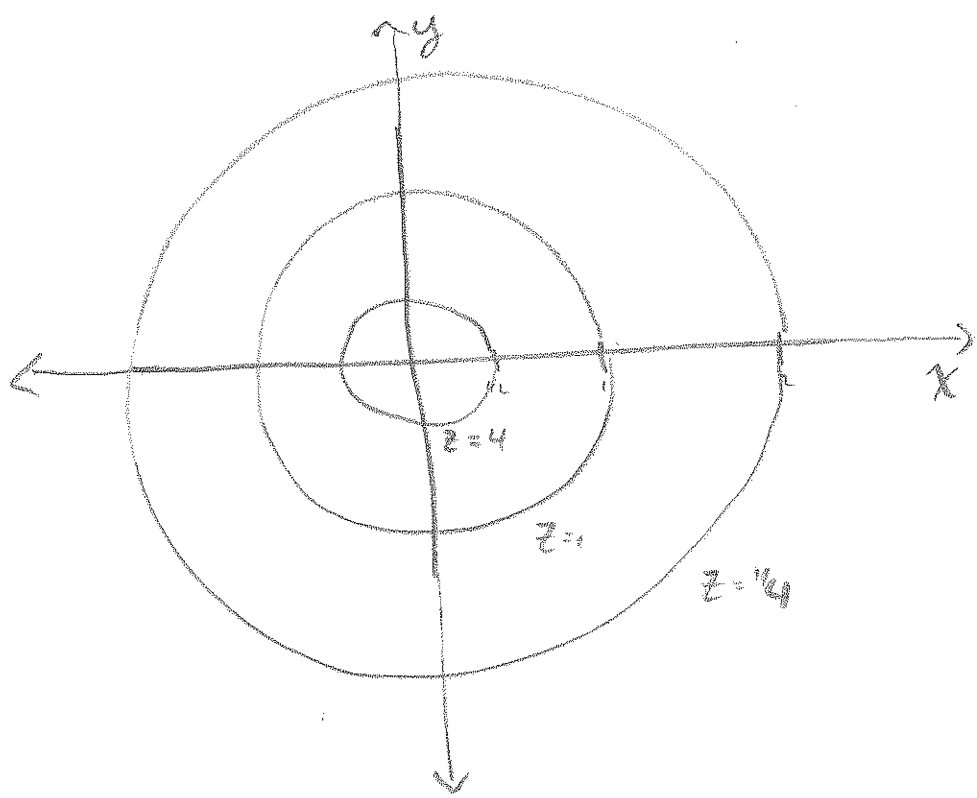
$$z = \frac{1}{x^2+y^2}$$

$$x^2 + y^2 = \frac{1}{z} \quad \leftarrow \text{Circle radius } \tfrac{1}{\sqrt{z}}$$

$z = 1/4 \Rightarrow x^2+y^2 = 4$ ← Radius 2

$z = 1 \Rightarrow x^2+y^2 = 1$ Radius 1

$z = 4 \Rightarrow x^2+y^2 = 1/4$ Radius 1/2



4. (15 points) Calculate the following limits

(a) $\displaystyle\lim_{(x,y)\to(-3,3)} \frac{x^2-y^2}{x+y}$ Plug-in $\frac{9-9}{-3+3} = 0/0$

$$= \lim_{(x,y)\to(-3,3)} \frac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}} = \lim_{(x,y)\to(-3,3)} x-y = -3-3$$
$$= \boxed{-6}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ Plug-in $0/0$ Try line $y=mx$

$$\lim_{(x,mx)\to(0,0)} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x^2}{x^2}\left(\frac{1-m^2}{1+m^2}\right)$$
$$= \frac{1-m^2}{1+m^2}$$

Since ↗ depends on m, (ie depends on path) $\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ DNE}}$

(c) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3-xy^2}{x^2-2xy+y^2}$ Plug-in $0/0$

$$= \lim_{(x,y)\to(1,1)} \frac{x(x^2-y^2)}{(x-y)^2} = \lim_{(x,y)\to(1,1)} \frac{x(x+y)\cancel{(x-y)}}{(x-y)^2}$$

$$= \lim_{(x,y)\to(1,1)} \frac{x(x+y)}{x-y} \quad \text{Plug-in} = \frac{1(2)}{0} = \frac{2}{0} \quad \boxed{\text{DNE}}$$

(In fact, there is an assymptote here)

5. Every month, a bakery produces 20 thousand loaves of white bread and 30 thousand loaves of wheat bread. The bakery's monthly profit is 23 thousand dollars. It estimates that increasing monthly production of white loaves by 1 thousand will increase monthly profit by 1 thousand dollars; it estimates increasing monthly production of wheat loaves by 1 thousand decreases monthly profit by 500 dollars.

   (a) (7 points) Using linear approximation, estimate the monthly profit if the bakery increases production to 22 thousand white loaves and 32 thousand wheat loaves per month.

Let x = # white loaves (in thousands)
y = # wheat loaves (in thousands)
$P(x,y)$ = Profit (in thousands)

$$P(20,30) = 23$$

$$\frac{\partial P}{\partial x}(20,30) = 1 \qquad \frac{\partial P}{\partial y}(20,30) = -\tfrac{1}{2}$$

$$L(x,y) = P(20,30) + \frac{\partial P}{\partial x}(20,30)(x-20) + \frac{\partial P}{\partial y}(20,30)(y-30)$$
$$= 23 + (x-20) - \tfrac{1}{2}(y-30)$$

$$L(22,32) = 23 + (22-20) - \tfrac{1}{2}(32-30)$$
$$= 23 + 2 - 1 = \boxed{24}$$

New monthly profit approx \$24,000.

   (b) (3 points) How should the bakery change their production levels to most efficiently increase profit (i.e. should the increase or decrease white/wheat production and in what ratio)?

Direction of greatest increase for P is in direction of ∇P.

$$\nabla P = \left\langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right\rangle \Big|_{(20,30)} = \langle 1, -\tfrac{1}{2} \rangle$$

It should increase white, decrease wheat, and make 1 less wheat loaf for every 2 new white loaves.

6. (10 points) Suppose that $f(u,v)$ is a differentiable function on $\mathbb{R}^2$ with $u = x + y$ and $v = x - y$. Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 2\frac{\partial f}{\partial u}.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u} 1 + \frac{\partial f}{\partial v} 1$$

$$= \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial f}{\partial u}\cdot 1 + \frac{\partial f}{\partial v}(-1)$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}$$

$$\boxed{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} + \cancel{\frac{\partial f}{\partial v}} + \frac{\partial f}{\partial u} - \cancel{\frac{\partial f}{\partial v}} = 2\frac{\partial f}{\partial u}}$$

7. Let $z = f(x, y) = 5y - 10x + \cos x$.

(a) (8 points) Find the equation of the tangent plane to f at the point $(x_0, y_0) = (0, 1)$.

$f_x = -10 - \sin x \quad \Longrightarrow \quad f_x(0,1) = -10$

$f_y = 5 \quad\quad\quad\quad\quad\quad f_y(0,1) = 5$

$f(0,1) = 5 - 0 + \cos 0 = 6$

Tang Plane $\quad z = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)$

$$\boxed{z = 6 - 10x + 5(y-1)}$$

(b) (5 points) Find the derivative of f at the point $(0, 1)$ in the direction of the vector $\langle -3, 4\rangle$.

If $\vec{v} = \langle -3, 4\rangle$, Then unit vector $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -3,4\rangle}{\sqrt{3^2+4^2}} = \langle -3/5, 4/5\rangle$

$D_{\vec{u}} f\Big|_{(0,1)} = \nabla f \cdot \vec{u}\Big|_{(0,1)} = \langle -10, 5\rangle \cdot \langle -3/5, 4/5\rangle$

$= 30/5 + 20/5 = \boxed{10}$

(c) (5 points) If $x(t) = -3t$, $y(t) = e^{4t}$, find $\frac{df}{dt}$ at $t = 0$.

$x(0) = -3(0) = 0$

$y(0) = e^0 = 1 \quad \Rightarrow$ Point $(0,1)$

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$\frac{df}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x}(0,1)(-3) + \frac{\partial f}{\partial y}(0,1)\, 4e^{4t}\Big|_{t=0}$

$= (-10)(-3) + 5 \cdot 4 = 30 + 20 = \boxed{50}$

8. (15 points) Let $f(x, y) = x^2 + xy + y^2 - 6x$.

(a) Find and classify all critical points of f.

(b) Find the absolute maximum and minimum values of f on the closed triangle with vertices $(0, 0)$, $(4, 0)$, and $(4, 4)$.

a. $\begin{cases} ①\ 0 = \partial f/\partial x = 2x + y - 6 \\ ②\ 0 = \partial f/\partial y = x + 2y \end{cases} \Rightarrow \underline{x = -2y}$

Plug in $x = -2y$ to eqn ①

$0 = 2(-2y) + y - 6 = -4y + y - 6$

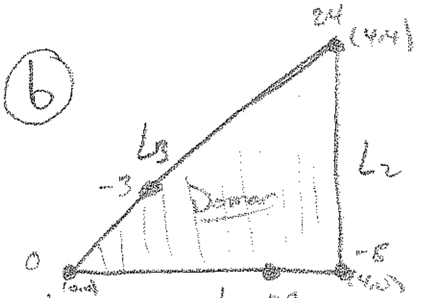
$0 = -3(y+2) \Rightarrow \underline{y = -2}$

$x = -2y = -2(-2) = \underline{4} \Rightarrow \underline{\text{Crit pt } (4, -2)}$

$f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3 > 0 \Rightarrow$ Local Extrema.

$f_{xx} > 0 \Rightarrow$ Conc Up $\Rightarrow$ $\boxed{(4, -2) \text{ is local min}}$

(b)



No critical points in domain, so we only need to check boundary curves.

L_1: $y = 0$, $0 \le x \le 4$

$f = x^2 - 6x$

$0 = f' = 2x - 6$

$x = 3$

$f(3) = 3^2 - 6 \cdot 3 = -9$

L_3: $y = x$, $0 \le x \le 4$

$f = x^2 + x^2 + x^2 - 6x$

$= 3x^2 - 6x$

$0 = f' = 6x - 6$

$\underline{x = 1}$

$f(1) = 3 - 6 = -3$

L_2: $x = 4$ $\quad 0 \le y \le 4$

$f = 16 + 4y + y^2 - 24$

$= -8 + 4y + y^2$

$0 = f' = 4 + 2y \Rightarrow y = -2$, not in Domain

Points to check

cp's on boundary curves $\begin{cases} f(3, 0) = -9 \leftarrow \text{Abs Min} \\ f(1, 1) = -3 \end{cases}$

Endpts of bound curves $\begin{cases} f(0, 0) = 0 \\ f(4, 0) = -8 \\ f(4, 4) = 24 \leftarrow \text{Abs Max} \end{cases}$