REVIEW OF INTEGRATION: MATH 234

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Warning: This is summary is only intended to help people who understand most of integration but get confused by the various kinds of integrals. If you do not yet know how to perform the integrals from chapters 15 and 16, this document will not help you. It is also not a comprehensive review. **Memorizing formulas without understanding how to apply them will not help you.**

1. Ordinary Integrals

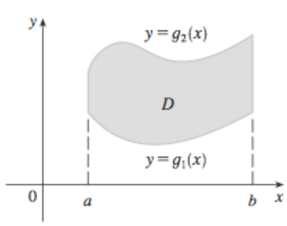
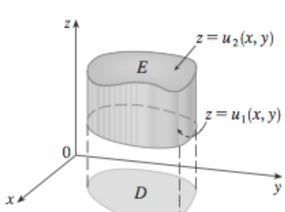
In Calculus I, you learned how to integrate over 1-dimensional subsets of $\mathbb{R}$, usually given by a closed interval $[a, b]$. In Chapter 15, you learned how to integrate over 2-dimensional subsets of $\mathbb{R}^2$ and 3-dimensional subsets of $\mathbb{R}^3$. More concisely, you learned how to integrate functions over n-dimensional subsets of $\mathbb{R}^n$, where $n = 1, 2, 3$. Note that we write one integral sign $\int$ for every dimension of the region of integration, hence the notations $\int$, $\iint$, and $\iiint$.

The key to performing these integrals is to find bounds of integration. For single integrals, this is simple. For double integrals, you must think about your region having a "shadow" $[a, b]$ in the x- or y-axis. For every point in the shadow, you find the bounds of integration in the perpendicular direction; these will usually depend on the point in the shadow. In the table below, these bounds are $g_1(x)$ and $g_2(x)$.

Finding the bounds of integration for a triple integral is the same process, but you have one extra dimension to deal with. Your 3-dimensional region has a shadow, which we call D, in the xy, xz, or yz plane. You then find the bounds of integration perpendicular to every point in the shadow. The shadow D is a 2-dimensional region, so you can use the previous paragraph to find its bounds.

The function being integrated has no relationship with the functions used to give the bounds of integration. The following table summarizes all this:

Integrals over n-dimensional subspaces of $\mathbb{R}^n$

n	Sample Region	Integral	Sample iterated integral
$n=1$		$\int_I f(x)\,dx$	$\int_{x=a}^{x=b} f(x)\,dx$
$n=2$		$\iint_D f(x, y)\,dA$	$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\,dy\,dx$
$n=3$		$\iiint_E f(x, y, z)\,dV$	$\iint_D \left[\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)\,dz\right] dA$

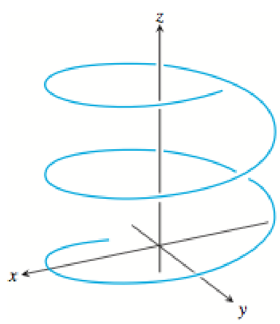
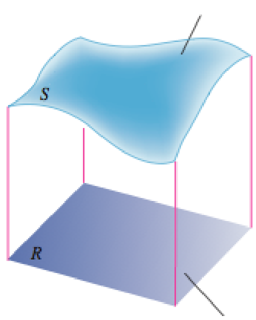
2. Line and Surface Integrals

A **line integral** is any integral where the region of integration is a 1-dimensional subspace of $\mathbb{R}^3$. This is different than Calculus I, where we only considered 1-dimensional subspaces of $\mathbb{R}$. A **surface integral** is any integral where the region of integration is a 2-dimensional subspace of $\mathbb{R}^3$; double integrals from Chapter 15 only involved 2-dimensional subspaces of $\mathbb{R}^2$.

A 1-dimensional subspace in $\mathbb{R}^3$ is a curve in space, so we call it C. Any curve C can be described as the image of a vector-valued function $\mathbf{r}(t)$, $a \le t \le b$. Such a function $\mathbf{r}(t)$ is called a parameterization. Any 2-dimensional subspace of $\mathbb{R}^3$, or surface S, can be given (at least locally) as the solution to an equation $g(x, y, z) = 0$. **To perform a line integral, you first need to parameterize your curve via $\mathbf{r}(t)$. To perform a surface integral, you must first find $g(x, y, z)$.**

Once you have your $\mathbf{r}$ or g, you use this to reduce the line/surface integral to an ordinary single/double integral. We just described how to compute ordinary single or double integrals in the previous section. For a line integral, the ordinary single integral will occur over the 1-dimensional subspace $[a, b]$ of $\mathbb{R}$. For a surface integral, the double integral will occur over the "shadow" of the surface S in either the xy, xz, or yz plane; this is a 2-dimensional subspace of $\mathbb{R}^2$. The integrand is transformed according to the equations below. **Below, the function f and vector field $\mathbf{F}$ are not related to the curve C or surface S; they are independent from $\mathbf{r}$ and g.**

Integrals over n-dimensional subspaces of $\mathbb{R}^3$

n	Sample Region	Algebraic description of region	Integral	Integral rewritten
$n = 1$		$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ $a \le t \le b$	$\int_C f\,ds$	$\int_{t=a}^{t=b} f \left\lvert \frac{d\mathbf{r}}{dt} \right\rvert dt$
			$\int_C \mathbf{F} \cdot d\mathbf{r}$	$\int_{t=a}^{t=b} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt$
$n = 2$		$g(x, y, z) = c$ S has shadow R in plane $\perp \mathbf{p}$	$\iint_S f\,d\sigma$	$\iint_R f \frac{\lvert\nabla g\rvert}{\lvert\nabla g \cdot \mathbf{p}\rvert} dA$
			$\iint_S \left(\mathbf{F} \cdot \mathbf{n}\right) d\sigma$	$\iint_R \frac{\mathbf{F} \cdot \pm\nabla g}{\lvert\nabla g \cdot \mathbf{p}\rvert} dA$

When you reduce a line integral to an ordinary single integral, make sure you use your parameterization to rewrite all variables x, y, z in terms of t. When you reduce a surface integral to an ordinary double integral, use your equation $g(x, y, z) = c$ to rewrite one of the variables in terms of the other two; e.g. if R is in the xy-plane, z should be written in terms of x and y.

The arclength and surface area correspond to $\int_C ds$ and $\iint_S d\sigma$ respectively. Calculating work or circulation of a vector field $\mathbf{F}$ along a curve C is $\int_C \mathbf{F}\cdot d\mathbf{r}$. The flux of $\mathbf{F}$ on a surface S is $\iint_S (\mathbf{F}\cdot\mathbf{n})\,d\sigma$. In the preceding two integrands, the dot product will change two vectors into a function, which is then integrated. Finally, the following two integrals are equal:

$$\int_C M\,dx + N\,dy + P\,dz = \int_C \langle M, N, P\rangle \cdot d\mathbf{r}.$$

It is just a notational difference. (You can also directly use the form on the left and substitute $dx \to \frac{dx}{dt}dt$ when writing everything in terms of t.)

3. FTC, Green, Stokes, Divergence

The Fundamental Theorem of Calculus for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem all say that two integrals are equal. In one integral, the integrand involves a sort of "derivative" and the region is an arbitrary n-dimensional region in $\mathbb{R}^3$. The region of integration for the other integral is the boundary of the previous region (so is $n-1$-dimensional); the integrand involves the original function or vector field before the "derivative." Note that we can think of evaluating f at two points as integrating over a 0-dimensional region.

In order to use any of these theorems, you must first understand what integral the problem wants you to figure out. Then, you use the theorem to transform the problem into a completely different integration problem. Once you have your new integral, use existing techniques to set it up and compute it.

Important Theorems

Regions	Integral 1		Integral 2
C is curve going from A to B	$\int_C (\nabla f)\cdot d\mathbf{r}$	$=$	$f\Big\vert_A^B = f(B) - f(A)$
D is region in $\mathbb{R}^2$ with boundary C	$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$	$=$	$\int_C M\,dx + N\,dy$
S is surface with boundary curve C	$\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,d\sigma$	$=$	$\int_C \mathbf{F}\cdot d\mathbf{r}$
E is region with boundary surface S	$\iiint_E (\nabla\cdot\mathbf{F})\,dV$	$=$	$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$

In theory, you could always compute the integral on either side of the equals sign. In practice, one of the integrals is often much simpler than the other. Each of these integrals is something that was discussed in Section 1 or 2 of this document.