Λ-modules

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1 Motivation

1.1 Background

Iwasawa considers such field extension tower:Let $K_n = Q(\zeta_{p^{n+1}})$

$$Q \subset K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset K_\infty = \cup K_n$$

We know that $Gal(K_n/Q) = (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ and

$$Gal(K_\infty/Q) = \mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \oplus \mathbb{Z}_p$$

$$Gal(K_n/K_0) = \Gamma/\Gamma^{p^n} \triangleq \Gamma_n$$

To understand such field extension, it is natural to consider the group $\mathbb{Z}_p[\Gamma_n]$ and how it acts on other stuff.

1.2 Construction

In fact, we do not only consider $\mathbb{Z}_p[\Gamma_n]$, we consider its inverse limit.

Let $\gamma \in \Gamma$ be the topological generator. If $m \geq n \geq 0$ there is a natural map $\phi_{m,n} : \mathbb{Z}_p[\Gamma_m] \to \mathbb{Z}_p[\Gamma_n]$ induced by the map $\Gamma_m \to \Gamma_n$

$$\mathbb{Z}_p[\Gamma_n] \cong \mathbb{Z}_p[T]/((1+T)^{p^n} - 1)$$

We take the inverse limit

$$\mathbb{Z}_p[[\Gamma]] \triangleq \varprojlim \mathbb{Z}_p[\Gamma_n] \cong \varprojlim \mathbb{Z}_p[T]/((1+T)^{p^n} - 1)$$

Clearly, $\mathbb{Z}_p[\Gamma] \subset \mathbb{Z}_p[[\Gamma]]$, but they are different. In fact, $\mathbb{Z}_p[[\Gamma]]$ is compactification of $\mathbb{Z}_p[\Gamma]$

Theorem 1 $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$, *the isomorphism being induced by* $\gamma \to 1 + T$

2 Structure of Λ-modules

Let $\Lambda = \mathbb{Z}_p[[T]]$, we first discuss some property of Λ.

Definition 2 *$P(T) \in \mathbb{Z}_p[T]$ is called distinguished if $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ with $a_i \in p\mathbb{Z}_p$*

Theorem 3 *(p-adic Weierstrass preparation theorem)Let*

$$f(T) = \sum_{i=0}^{\infty} a_i T^i \in \Lambda$$

and assume for some n we have $a_i \in p\mathbb{Z}_p, 0 \leq i \leq n-1$, but $a_i \notin p\mathbb{Z}_p$. Then f can be uniquely written in the form $f(T) = P(T)U(T)$, where$U(T) \in \Lambda$ is a unit and $P(T)$ is a distinguished polynomial of degree n.

More generally, Let $f(T)\ in \Lambda$, then we may write

$$f(T) = p^{\mu} P(T)U(T)$$

where P and U as above and μ is a nonnegative integer.

Theorem 4 *(Division algorithm) If $F(T) \in \Lambda$ and $P(T)$ is distinguished then uniquely*

$$f(T) = q(T)P(T) + r(T)$$

with $r(T) \in \mathbb{Z}_[T]$, deg $r(T)$ <deg$P(T)$.

Now, we can conclude that Λ is a UFD. In fact, Λ is a noetherian regular local ring with Krull dimension 2. The height 0 prime is 0, the height 1 prime is (p) and $P(T)$ where $P(T)$ is irreducible and distinguished, the height 2 prime is (p, T) which is the unique maximal ideal.

Lemma 5 *Suppose $f, g \in \Lambda$ are relatively prime. Then the ideal (f,g) is of finite index in Λ*

Lemma 6 *Suppose $f, g \in \Lambda$ are relatively prime. Then*

- *the natural map*
$$\Lambda/(fg) \to \Lambda/(f) \oplus \Lambda/(g)$$
is an injection with finite cokernel

- *there is an injection*
$$\Lambda/(f) \oplus \Lambda/(g) \to \Lambda/(fg)$$
with finite cokernel

Definition 7 *Two Λ modules M and M' are said to be pseudo-isomorphic, written*

$$M \sim M'$$

if there is an exact sequence of Λ modules

$$0 \to A \to M \to M' \to B \to 0$$

Warning: $M \sim M'$ does not imply $M' \sim M$. For example, $(p, T) \sim \Lambda$

Remark 8 *For finitely generated Λ modules,*

$$M \sim M' \Leftrightarrow M' \sim M$$

Theorem 9 *Let M be a finitely generated Λ module. Then*

$$M \sim \Lambda^r \oplus (\bigoplus_{i=1}^{s} \Lambda/(p^{n_i})) \oplus (\bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{m_j}))$$

where$r, s, t, n_i, m_j \in \mathbb{Z}$, and f_j is distingushied and irreducible.

Proof. Localization or row and column operation ■

It is first proved by Iwasawa in terms of the group ring $\mathbb{Z}_p[[\Gamma]]$. Serre observed that the group ring is isomorphic to Λ and deduced the structure theorem from some general results in commutative algebra.Paul Cohen showed that one could give a proof via row and column operation.

3 Adjoints

Let X be a finitely generated torsion Λ module. We know X will be pseudo-isomorphic to an "elementary' Λ module

$$E = \oplus_i \Lambda/(f_i^{m_i})$$

where f_i can be p or an irreducible distinguished polynomial. we define the characteristic polynomial of X to be

$$char(X) = \prod f_i^{m_i}$$

We want to show such E is uniquely determined by X

Lemma 10 *Let $X \sim \oplus\Lambda/(f_i^{m_i})$ as above. Then*

$$X \otimes_\Lambda \Lambda_\mathfrak{p} = \oplus_{(f_i)=\mathfrak{p}} \Lambda_\mathfrak{p}/f_i^{m_i}\Lambda_\mathfrak{p}$$

where $\mathfrak{p}$ is height prime

Corollary 11 *X is finite if and only if $X \otimes \Lambda_\mathfrak{p} = 0$ for all height one prime*

Lemma 12 *Let $\psi : X \to \oplus_\mathfrak{p}(X \otimes \Lambda_\mathfrak{p})$ be the natural map. Then $Ker\psi$ is finite and is the maximal finite submodule of X*

Now we know how to characterize the $ker\psi$, how about $Coker\psi$? Before we describe $Coker\psi$, we define the adjoint of X

Definition 13 *Define*

$$\tilde{\alpha}(X) = Hom_{\mathbb{Z}_p}(Coker\psi, \mathbb{Q}_p/\mathbb{Z}_p)$$

the action of Λ is given by $(\gamma f)(x) = f(\gamma^{-1}x)$ for $\gamma \in \Gamma$ and $x \in Coker\psi$. Hence,$(g(T)f)(x) = f(g((1+T)^{-1}-1)x)$ for $g(T) \in \Lambda$

Inspired by such action, for any Λ module X, we can define a new action of Λ on X by

$$\gamma * x = \gamma^{-1}x$$

Let $\tilde{X}$ be X with the new action. Then the adjoint of X is

$$\alpha(X) = \tilde{\alpha}(\tilde{X})$$

Now we will compute the adjoint of X. First, we need to describe $Coker\psi$

Define an admissible sequence to be a sequence $\sigma_0, \sigma_1, \cdots$ of elements of Λ such that σ_n and Char(X) are relatively prime, and $\sigma_{n+1}/\sigma_n \in (p, T)$ for all $n \geq 0$. Note that

$$\frac{1}{\sigma_0}\Lambda \subset \frac{1}{\sigma_1}\Lambda \subset \frac{1}{\sigma_2}\Lambda \subset \cdots$$

and

$$\varinjlim \frac{1}{\sigma_n}\Lambda = \cup \frac{1}{\sigma_n}\Lambda$$

Proposition 14 *The map*

$$\phi : X \otimes_\Lambda (\cup \frac{1}{\sigma_n}\Lambda) \to \oplus_{\mathfrak{p}}(X \otimes_\Lambda \Lambda_{\mathfrak{p}})$$

$$x \otimes \frac{1}{\sigma_n} \to (\cdots, x \otimes \frac{1}{\sigma_n}, \cdots)$$

is an isomorphism of Λ modules(the direct sum is over any set of $\mathfrak{p}$ containing all (height one) prime divisors of char(X) and such that $\sigma_n \in \Lambda_{\mathfrak{p}}^\times$ for all n and $\mathfrak{p}$)

Applying $X \otimes_\Lambda$ to the exact sequence

$$0 \to \Lambda \to \cup \frac{1}{\sigma_n}\Lambda \to (\cup \frac{1}{\sigma_n}\Lambda)/\Lambda \to 0$$

yields

$$X \to \oplus_{\mathfrak{p}}(X \otimes_\Lambda \Lambda_{\mathfrak{p}}) \to X \otimes (\cup \frac{1}{\sigma_n}\Lambda)/\Lambda \to 0$$

Therefore

$$Coker\psi \cong X \otimes_\Lambda (\cup \frac{1}{\sigma_n}\Lambda)/\Lambda$$

In fact, we can choose special σ_n to make this more explicitly. For example, let $\sigma_n = (T-\pi)^n$ with $\pi \in p\mathbb{Z}_p$then

$$\cup \frac{1}{\sigma_n}\Lambda = \Lambda[\frac{1}{T-\pi}]$$

which is the ring of Laurent series.

Theorem 15 *Assume $f \in \Lambda, \pi \in p\mathbb{Z}_p$ and $f(\pi) \neq 0$. Then*

$$\Lambda/(f) \cong Hom_{\mathbb{Z}_p}(\Lambda/(f) \otimes \widetilde{\Lambda[\frac{1}{T-\pi}]}/\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) \cong \alpha(\Lambda/(f))$$

Proof. For $g = \sum_{i=-N}^{\infty} a_i (T-\pi)^i$ with $a_i \in \mathbb{Q}_p$, define $Res_{T=\pi} g = a_{-1}$. Define a pairing

$$\Lambda/(f) \times [\Lambda/(f) \otimes \Lambda[\frac{1}{T-\pi}]/\Lambda] \to \mathbb{Q}_p/\mathbb{Z}_p$$

$$(a, b \otimes c) = Res_{T=\pi}(\frac{abc}{f})(mod \mathbb{Z}_p)$$

■

Lemma 16 *Suppose A and B are $\mathbb{Z}_p$ modules with $A \cong \mathbb{Z}_p^n$. Assume there is a nondegenerate pairing*

$$A \times B \to \mathbb{Q}_p/\mathbb{Z}_p$$

then $A \cong Hom_{\mathbb{Z}_p}(B, \mathbb{Q}_p/\mathbb{Z}_p)$

Corollary 17 *If E is an elementary torsion Λ module, then $E \cong \alpha(E)$.*

Proposition 18 *Let X and Y be finitely generated torsion Λ modules with $X \sim Y$, then $\alpha(Y) \sim \alpha(X)$*

Corollary 19 *$X \sim \alpha(X)$, so $\alpha(X)$ us also finitely generated torsion Λ module.*