Iwasawa λ invariant and Massey products

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Main theorem

In some cases, the λ invariant can be computed in terms of Massey products.

Iwasawa theory

Let $K \subset K_1 \subset K_2 \subset \cdots \subset K_l \subset \cdots \subset K_\infty$ be a $\mathbb{Z}_p$ extension of number field K,i.e $\mathrm{Gal}(K_l/K) \cong \mathbb{Z}/p^l\mathbb{Z}$ and $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$.

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Theorem (Iwasawa[Iwa59])

There are constants μ, λ, ν such that when l is sufficient large,

$$\#\mathrm{Cl}(K_l)[p^\infty] = p^{\mu p^l + \lambda l + \nu}$$

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Remark

In the case of interests of the talk, it is known that $\mu = 0$, so the most interesting invariant is λ.

Gold's criterion

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- *Then*

$$\lambda \geq 2 \iff \chi \cup \alpha = 0$$

Here α *is a generator of* $\mathfrak{P}_0^{h_K}$ *and* χ *is an element in* $H^1(\mathrm{Gal}(K_S/K), \mathbb{Z}_p)$*.*

McCallum-Sharifi's result

Let μ_n be the group of n-th roots of unity.

Theorem (McCallum-Sharifi[MS03])

- *Let* $K = \mathbb{Q}(\mu_p) \subset \mathbb{Q}(\mu_{p^2}) \subset \cdots \subset \mathbb{Q}(\mu_{p^l}) \subset \cdots \subset \mathbb{Q}(\mu_{p^\infty})$ *be a cyclotomic* $\mathbb{Z}_p$ *extension.*

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- *Decompose* $\mathrm{Cl}(\mathbb{Q}(\mu_p))[p^\infty] = \oplus_i \varepsilon_i \mathrm{Cl}(\mathbb{Q}(\mu_p))[p^\infty]$ *as direct sum of eigenspaces (pieces) with respect to the action of* $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$.

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- *Decompose* $\mathrm{Cl}(\mathbb{Q}(\mu_p))[p^\infty] = \oplus_i \varepsilon_i \mathrm{Cl}(\mathbb{Q}(\mu_p))[p^\infty]$ *as direct sum of eigenspaces (pieces) with respect to the action of* $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$.
- *Assume conditions. Then*

$$\lambda_i \geq 2 \iff \chi \cup \alpha_i = 0$$

Where α_i *is an element* K^* *constructed from* i*-th piece and* λ_i *is the Iwasawa invariant that corresponds to the* i*-th piece.*

Comparing

Theorem (Gold's criterion[Gol74])

Let K be an imaginary quadratic field. For cyclotomic $\mathbb{Z}_p$ extensions, under some conditions:

$$\lambda \geq 2 \Longleftrightarrow \chi \cup \alpha = 0$$

Theorem (McCallum-Sharifi[MS03])

Let K be a cyclotomic field $\mathbb{Q}(\mu_p)$. For cyclotomic $\mathbb{Z}_p$ extensions, under some conditions:

$$\lambda_i \geq 2 \Longleftrightarrow \chi \cup \alpha_i = 0$$

for odd $i > 1$.

Remark

Both theorems has the form "$\lambda \geq 2 \Longleftrightarrow \chi \cup \alpha = 0$", which motivates us to find the deep reason behind it.

Cup product is obstruction for us to glue

Slogan

Massey product is a generalization of cup products.

- Given $\chi_1, \chi_2 \in H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p)$, we can form two representations $G \to GL_2(\mathbb{F}_p)$:

$$\rho_{\chi_1}(g) = \begin{pmatrix} 1 & \chi_1(g) \\ 0 & 1 \end{pmatrix}, \rho_{\chi_2}(g) = \begin{pmatrix} 1 & \chi_2(g) \\ 0 & 1 \end{pmatrix}$$

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- Try to glue the two representations together:

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- Try to glue the two representations together:

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- We want to fill $*$ spot a cochain $\phi \in \mathcal{C}^1(G, \mathbb{F}_p)$ such that the above is a representation.

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- We can fill $*$ spot a cochain $\phi \in \mathcal{C}^1(G, \mathbb{F}_p)$ s.t. the above is a representation $\iff \chi_1 \cup \chi_2 = -d\phi$ in $\mathcal{C}^\cdot(G, \mathbb{F}_p) \iff \chi_1 \cup \chi_2 = 0$ in $H^2(G, \mathbb{F}_p)$.

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- Cup product $\chi_1 \cup \chi_2$ is the obstruction for us to glue.
- Generally if we have higher dimensional representations derived from elements in $H^1(G, \mathbb{F}_p)$ and they are compatible in a certain way, Massey products are the obstruction for us to glue them.

Massey products and knots

- Massey products were first introduced by considering the following knots.
- Analogy between knots and primes:

$$\text{knot } S^1 \hookrightarrow \mathbb{R}^3 \longleftrightarrow \text{prime } \mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$$

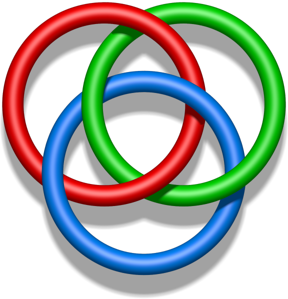

Figure: Borromean Rings

Main result

Theorem (Q.)

- *Let* $K \subset K_1 \subset K_2 \subset \cdots \subset K_\infty$ *be a* $\mathbb{Z}_p$ *extension of* K
- *Let* S *be the set of primes above* p *for* K
- K_∞/K *is totally ramified for all primes in* S.
- *Let* $X = \varprojlim \mathrm{Cl}_S(K_l)$ *and* μ, λ *be the Iwasawa invariant of* X.
- *Assume* X *has no torsion element and* $H^2(G_{K,S}, \mu_p) \cong \mathbb{F}_p$.

Then $\mu = 0$ *if and only if there exists* k *such that the generalized Bockstein map* $\Psi^{(k)} \neq 0$ *for some* k. *If* $\mu = 0$, *then* $\lambda = min\{n|\Psi^{(n)} \neq 0\} - \#S + 1$

Applying to the previous cases

Corollary (Q.)

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- *Let K be an imaginary quadratic field and the same setting up as Gold's criterion.*
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Corollary (Q.)

- *Let $K = \mathbb{Q}(\mu_p)$ and the same setting up as McCallum-Sharifi's result.*
- *Then* $\lambda_i = \min\{n|n$ *fold "Massey product"* $\varepsilon_i(\chi, \chi, \cdots, \chi, \alpha_i)$ *is nonzero*$\}$.

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- Some classical results tells us "$\lambda \geq 2 \Longleftrightarrow$ certain cup product vanishes" under some conditions.
- Massey product is a generalization of cup products.
- Main results: " Assume $\lambda \geq n-1$, then $\lambda \geq n \Longleftrightarrow$ certain Massey product vanishes" under some conditions.

THANK YOU!

References I

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