A short introduction to continuous dependence results for Hamilton-Jacobi equations

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1 Introduction

The Hamilton-Jacobi equations,

$$u_t + H(t, x, u, \nabla u) = 0 \qquad \text{in } \mathbf{R}^d,$$

arise in many areas of applied mathematics like optimal control [22], differential games [9], seismic wave propagation [23, 21, 18], terrain navigation (computation of minimum time transit paths) of robots [3], and financial mathematics [10] among many others. They also appear when modeling evolving interfaces in geometry, fluid mechanics, computer vision [19, 13], and materials science [20]; and are essential when dealing with level set methods [17, 15], which are numerical methods that have reached a widespread popularity nowadays. In this paper, we provide a *short introduction* to the techniques that allow us to obtain continuous dependence results for these equations.

Since it is impossible to carry out this study for the above Hamilton-Jacobi equation in this paper, we are going to restrict ourselves to the following steady-state

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model Hamilton-Jacobi equation

$$u + H(\nabla u) = f,$$

where u and f are periodic in each space coordinate with period 1, since it is not difficult to extend the results for this model Hamilton-Jacobi equation to the general case. This will allow us to provide a shorter, simpler and more clear presentation of the main ideas associated with continuous dependence results for these equations.

The continuous dependence results we consider are of the form

$$\| u - v \| \le \Phi(v), \tag{1}$$

where $\| \cdot \|$ is a norm, u is the exact solution and v is an (almost) arbitrary function. Note that, if v is an approximation to the exact solution, the estimate (1) gives a rigorous measure of the quality of the approximation; these estimates, called *a posteriori* error estimates are thus very useful in practice.

They are also very useful to obtain theoretical properties of the exact solution. Indeed, since we expect to have that $\Phi(u) = 0$, it is reasonable to expect the functional $\Phi(v)$ to depend solely on the *residual* of v, namely,

$$R(v) = v + H(\nabla v) - f.$$

Now, if v satisfies the model equation with $f = g$, the estimate (1) gives us a continuous dependence result with respect to the right-hand side, and if v satisfies the model equation with $H = \bar{H}$, the estimate (1) gives us a continuous dependence result with respect to the Hamiltonian. We thus see that important properties of the exact solution can be easily deduced from the continuous dependence result (1), as claimed.

We are also interested in studying continuous dependence results for approximations u_h to the exact solution. We get continuous dependence results of the form

$$\| u_h - v \|_h \le \Phi_h(v), \tag{2}$$

where $\| \cdot \|_h$ is a discrete version of the norm $\| \cdot \|$ and $\Phi_h(\cdot)$ a discrete version of $\Phi(\cdot)$. Note that if v is taken to be the exact solution of the Hamilton-Jacobi equation u, then the estimate (2) is an estimate of the quality of the approximation u_h to u which only depends on the exact solution u; these estimates, called *a priori* error estimates can tell us what kind of accuracy it is reasonable to expect from the approximation. Moreover, by using the above continuous dependence result, properties similar to those obtained for the exact solution can also be obtained for its approximation. Let us assume that the approximate solution u_h is defined by

$$u_h + \widehat{H}(\nabla_h u_h) = f,$$

where ∇_h is an approximation to ∇. Since we should have $\Phi_h(u_h) = 0$, we expect the functional $\Phi_h(v)$ to depend solely on the *truncation error* of v,

$$T(v) = v + \widehat{H}(\nabla_h v) - f.$$

Just as in the continuous case, from this estimate, continuous dependence results of the approximate solution u_h with respect to the initial data, the right-hand side and the approximate Hamiltonian $\bar{H}$ can be obtained.

In this paper, we obtain the continuous dependence result (1) for the exact solution. Then we show that for the so-called monotone schemes, continuous dependence results (2) can be obtained in a remarkably similar way.

Now, to obtain the continuous dependence result for the Hamilton-Jacobi equations, we could have used the elegant theory of viscosity solutions; see [7], [6]. However, we have chosen to follow a path we estimate to be more natural. We consider the parabolic problem,

$$u + H(\nabla u) - \nu \Delta u = f,$$

and obtain a continuous dependence result of the form

$$\| u - v \| \leq \Phi_\nu(v), \tag{3}$$

where v is a general function. Then, we show that when we let ν go to zero in (3), we obtain the continuous dependence result (1).

The main reasons that lead us to take this approach are that it emphasizes the fact that the viscosity solution is a limit of solutions of parabolic problems, and, accordingly, that the estimate (1) is *also* a limit of the estimate (3). An additional interesting consequence of this approach is that both the uniqueness of the viscosity solution as well as one of its classical characterizations can be *deduced* from the estimate (1). In this way, we emphasize that *to obtain continuous dependence results for the singular limit when $\nu \downarrow 0$, that is, for the viscosity solution, it is enough to obtain continuous dependence results for smooth solutions of the parabolic problem.* A similar idea was developed in [4] for hyperbolic scalar conservation laws.

The paper is organized as follows. In §2, we consider *smooth* solutions of the elliptic model problem and show how to find a very simple continuous dependence result of the type (3). We then show that although this result gives rise to many interesting properties it breaks down when $\nu \downarrow 0$. This motivates the modification presented in §3 which does not break down when $\nu \downarrow 0$; in this section, we follow [5]. In §4, we pass to the limit in ν to obtain the wanted estimate (1) for the viscosity solution of the steady-state Hamilton-Jacobi equation; the main result of this section was obtained, by different means, in and can be viewed as an extension of some of the results obtained in [6]. In §5, we introduce monotone schemes and obtain the estimate of the form (2) by mimicking the techniques used in the continuous case; this section contains a new unified approach to a priori error estimates which contains some of the error estimates for monotone schemes obtained in [8], [1], and in [12]. In §6, we end by giving some references for the reader interested in extensions of these results to more complicated Hamiltonians and to the time-dependent case.

2 A first continuous dependence result for elliptic equations

In this section, we begin our program by considering the problem of how to compare the solution of the elliptic problem

$$u + H(\nabla u) - \nu \Delta u = f \qquad \text{in } \mathbf{R}^d, \tag{4}$$

where u and f are periodic in each coordinate with period 1, with an arbitrary smooth function v which we also take to be periodic in each coordinate with period 1. We take both the Hamiltonian H and the right-hand side f to be $\mathcal{C}^\infty(\mathbf{R}^d)$ since it is known that for this data, a unique $\mathcal{C}^\infty(\mathbf{R}^d)$ solution u exists; see Friedman [11].

We obtain a remarkably simple continuous dependence result which allows us to obtain (i) a maximum principle for the exact solution, (ii) the so-called L^∞-contraction property for exact solutions, (iii) an a priori estimate on the $W^{1,\infty}(\Omega)$-semi-norm of the exact solution, (iv) a continuous dependence result of the exact solution with respect to the Hamiltonian H, and (v) an a posteriori error estimate.

Unfortunately, this continuous dependence result does *not* allow us to obtain a continuous dependence result with respect to the coefficient ν. Moreover, we show that the continuous dependence result *breaks down* as we let the coefficient ν go to zero; this continuous dependence result is *not* preserved under this limit process.

2.1 A simple continuous dependence result

The continuous dependence result we present gives an upper bound for the following semi-norms:

$$|\, u - v\,|_- = \sup_{x \in \Omega} (\, u(x) - v(x)\,)^+, \qquad |\, u - v\,|_+ = \sup_{x \in \Omega} (\, v(x) - u(x)\,)^+,$$

where $w^+ \equiv \max\{0, w\}$ and $\Omega = [0,1)^d$. An upper bound of the uniform norm can be easily obtained by noting that

$$\|\, u - v\,\|_{L^\infty(\Omega)} = \max_{\sigma \in \{-,+\}} |u - v|_\sigma . \tag{5}$$

The result is stated in terms of the *residual* of the function v, namely,

$$R(v; x) = v(x) + H(\nabla v(x)) - \nu \Delta v(x) - f(x). \tag{6}$$

We are now ready to state the result.

Theorem 1 (First Continuous Dependence Result for elliptic equations). *Let u be the solution of the equation (4) and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function periodic in each coordinate with period 1. Then, for $\sigma \in \{-,+\}$, we have that*

$$|\, u - v\,|_\sigma \le \Phi_\sigma(v), \tag{7}$$

where

$$\Phi_\sigma(v) = \sup_{x \in \mathbf{R}^d} \big(\sigma\, R(v; x)\big)^+. \tag{8}$$

It is important to point out the following consequences of this result:

a. The a posteriori error estimate. Note that, if v is considered to be an approximation to the exact solution u, Theorem 1 allows us to estimate the L^∞-error between u and v solely in terms of v. Since this can be done only *after* the computation of v, estimates like this are called *a posteriori error estimates*; they are very useful in practical situations as they give a rigorous upper bound of the error.

b. Dependence of the right-hand side: The maximum principle. Let us denote by u_f the solution of the equation (4) for a given Hamiltonian H. If we take $v = u_g$, since $R(v,x) = g(x) - f(x)$, Theorem 1 becomes

$$|\, u_f - u_g\,|_- \le \sup_{x\in\mathbf{R}^d} \big(f(x) - g(x)\big)^+, \qquad |\, u_f - u_g\,|_+ \le \sup_{x\in\mathbf{R}^d} \big(g(x) - f(x)\big)^+. \quad (9)$$

The above inequalities imply the maximum principle

$$-\sup_{x\in\Omega}(H(0) - f(x))^+ \le u_f(y) \le \sup_{x\in\Omega}(f(x) - H(0))^+ \qquad \forall\, y \in \Omega. \quad (10)$$

c. Dependence on the right-hand side: The $\mathbf{L^\infty}$-contraction property. The estimates (9) also imply the so-called L^∞-contraction property of the solutions of the equation (4), namely,

$$\|\, u_f - u_g\,\|_{L^\infty(\Omega)} \le \|\, f - g\,\|_{L^\infty(\Omega)}. \quad (11)$$

Note that by using this property, we can consider solutions of the equation (4) with $f \in \mathcal{C}^0(\mathbf{R}^d)$ only.

Also, if we take $g(x) = f(x + \delta p)$ for any given non-zero $p \in \mathbf{R}^d$, since $u_g(x) = u_f(x + \delta p)$, we get that

$$\frac{\|\, u_f(\cdot) - u_f(\cdot + \delta p)\,\|_{L^\infty(\Omega)}}{|\,\delta\,|} \le \frac{\|\, f(\cdot) - f(\cdot + \delta p)\,\|_{L^\infty(\Omega)}}{|\,\delta\,|},$$

and so, by taking the limit when $|\,\delta\,| \downarrow 0$ and since p is arbitrary, we obtain

$$|\, u_f\,|_{W^{1,\infty}(\Omega)} \le |\, f\,|_{W^{1,\infty}(\Omega)}. \quad (12)$$

Note that $|\cdot|_{W^{1,\infty}(\Omega)}$ is a semi-norm; indeed, if $|\, f\,|_{W^{1,\infty}(\Omega)} = 0$, then $f$is a constant. From the above inequality, this implies that u is also a constant; this is true since in this case, we have $u = f - H(0)$.

b. Dependence on the Hamiltonian. Let us denote by u_H the solution of the equation (4) for a given f. If we take $v = u_{\bar H}$ in Theorem 1, since $R(v,x) = H(\nabla v(x)) - \bar H(\nabla v(x))$, we get

$$\|\, u_H - u_{\bar H}\,\|_{L^\infty(\Omega)} \le \sup_{p\in\mathbf{R}^d:\|\,p\,\|\le|\,f\,|_{W^{1,\infty}(\Omega)}} \big|\, H(p) - \bar H(p)\,\big|. \quad (13)$$

Note that we have used the bound (12). By using this property, we can consider solutions of the equation (4) with $H \in \mathcal{C}^0(\mathbf{R}^d)$ only.

2.2 Proof of the First Continuous Dependence Theorem for Elliptic Equations

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. Define the auxiliary function

$$\psi(x) = u(x) - v(x),$$

and let $\widehat{x} \in \Omega$ be such that

$$\psi(\widehat{x}) \geq \psi(y) \qquad \forall y \in \Omega;$$

such a point exists since ψ is continuous and periodic on Ω. Moreover, since $\nabla\psi(\widehat{x}) = 0$, we can set $\widehat{p} = \nabla u(\widehat{x}) = \nabla v(\widehat{x})$.

We assume that $|\, u - v \,|_- > 0$, otherwise there is nothing to prove. In this case, we have

$$\begin{aligned}
|\, u - v \,|_- &= \sup_{x \in \Omega} \{u(x) - v(x)\} \\
&= \sup_{x \in \Omega} \psi(x) \\
&= u(\widehat{x}) - v(\widehat{x}) \\
&= \quad \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\quad - \big[v(\widehat{x}) + H(\widehat{p}) - \nu\Delta v(\widehat{x}) - f(\widehat{x})\big] \\
&\quad + \big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big] \\
&= R(u;\widehat{x}) - R(v;\widehat{x}) + \big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big],
\end{aligned}$$

by the definition of $\widehat{p}$ and that of the residual R, (6).

Now, since u is the exact solution of (4), $R(u;\widehat{x}) = 0$, and since $u - v$ attains a maximum at $\widehat{x}$,

$$\big[\nu\Delta u(\widehat{x}) - \nu\Delta v(\widehat{x})\big] \leq 0,$$

we get

$$|\, u - v \,|_- \leq (-R(v;\widehat{x}))^+,$$

and the result follows. This completes the proof of Theorem 1.

2.3 A key observation and the breaking down of the estimate

Although the stability Theorem 1 is remarkable in its simplicity, it is useless when we let the coefficient ν tend to zero. A key observation to understand why is this the case is to realize that the proof of Theorem 1 is, as a matter of fact, the proof of a *stronger* result. It is stated in terms of the functionals

$$\begin{aligned}
\|\, u - v \,\|_- &= \sup_{x:|\, u-v\,|_- = u(x)-v(x)} (\, u(x) - v(x)\,)^+ + \nu\big[-\Delta(u(x) - v(x))\big]^+, \\
\|\, u - v \,\|_+ &= \sup_{x:|\, u-v\,|_+ = v(x)-u(x)} (\, v(x) - u(x)\,)^+ + \nu\big[-\Delta(v(x) - u(x))\big]^+,
\end{aligned}$$

which, as we can see, are stronger than the functionals $|\,u-v\,|_-$ and $|\,u-v\,|_+$, respectively.

Theorem 2 (Strengthened First Continuous Dependence Result). *Let u be the solution of the equation* (4) *and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function periodic in each coordinate with period* 1. *Then, for $\sigma \in \{-,+\}$, we have that*

$$\|\!| u-v \|\!|_\sigma \le \Phi_\sigma(v), \tag{14}$$

where

$$\Phi_\sigma(v) = \sup_{x\in\mathbf{R}^d} \big(\sigma\, R(v;x)\big)^+. \tag{15}$$

Note that the above result shows that the functional $\Phi_\sigma(v)$ is measuring *more* than just the semi-norm $|\,u-v\,|_\sigma$ since it adds ν times the Laplacian of the error. This additional term is not necessarily of the same order as that of the semi-norm $|\,u-v\,|_\sigma$; in fact, it might be considerably bigger. This is precisely why the estimate of Theorem 1 (and that of Theorem 2) breakdown takes place when ν goes to zero.

To illustrate this phenomenon, let us consider the following simple but illuminating example. Consider the functions

$$\zeta_\nu(x) = -\nu \ln\big(\exp(x/\nu) + 2 + \exp(-x/\nu)\big),$$

which is the smooth solution of the parabolic equation

$$u + \frac{1}{2}(u')^2 - \nu\, u'' = f_\nu,$$

where $f_\nu(x) = \zeta_\nu(x) + 1/2$. Note that since $\lim_{\nu\downarrow 0} \zeta_\nu(x) = \zeta_0(x) = -|\,x\,|$, the second order derivative of ζ_ν at $x=0$ has to blow up as $\nu \downarrow 0$. This kink of ζ_0 is responsible for the breakdown of the stability result under consideration as we show next.

Although we cannot apply the above theorem since these functions are not periodic, it is not difficult to verify that Theorem 2 holds for $u=\zeta_\nu$ and $v=\zeta_{\bar\nu}$ for any positive parameters ν and $\bar\nu$. Moreover, in the case in which $\nu < \bar\nu$, a simple computation gives that

$$\|\!| \zeta_\nu - \zeta_{\bar\nu} \|\!|_+ = \Phi_+(\zeta_{\bar\nu}) = 0,$$

and that

$$|\,\zeta_\nu - \zeta_{\bar\nu}\,|_- = (\bar\nu - \nu)\ln(2), \qquad \|\!| \zeta_\nu - \zeta_{\bar\nu} \|\!|_- = \Phi_-(\zeta_{\bar\nu}) = (\bar\nu-\nu)\ln(2) + \frac{\bar\nu - \nu}{2\,\bar\nu},$$

where we have used the identity $\zeta''_\nu(0) = \frac{1}{2\nu}$ to compute the last term.

Note that the last equality shows that Theorem 2 is *sharp*. Note also that as a consequence of the above computations,

$$\lim_{\nu\downarrow 0} |\,\zeta_\nu - \zeta_{\bar\nu}\,|_- = |\,\zeta_0 - \zeta_{\bar\nu}\,|_- = \bar\nu\ln(2), \quad \text{and} \quad \lim_{\nu\downarrow 0} \Phi_-(\zeta_{\bar\nu}) = \bar\nu\ln(2) + \frac{1}{2}.$$

In other words, the result of Theorem 1 cannot give us any useful information about how close $\zeta_{\bar\nu}$ is to ζ_0. This is what we mean by the breakdown of this estimate when ν goes to zero.

3 A continuous dependence result that holds as $\nu \downarrow 0$

The results of last section show that we need to obtain a new continuous dependence result that is actually useful when $\nu \downarrow 0$. Our efforts will be then devoted to obtaining a *modification* of Theorem 1 that would allow us to compare solutions of our elliptic problem with different values of the parameter ν.

3.1 The new continuous dependence result

To state the ne continuous dependence result, we need to introduce two quantities. The first what we call the *generalized residual*, namely,

$$R_\epsilon^{\bar\nu}(u;x,p) = u(x) + H(p) - \bar\nu \Delta u(x) - f(x - \epsilon p) - \frac{\epsilon}{2}|p|^2. \qquad (16)$$

Note that, indeed, $R_\epsilon^{\bar\nu}$ is nothing but the residual R when $\epsilon = 0$ and $\bar\nu = \nu$. Note that we introduce this quantity because we are interested in dealing with functions v that might be solutions of the elliptic problem (4) with $\bar\nu \neq \nu$!

The second is the paraboloid P_v defined as follows:

$$P_v(x,p,\kappa;y) = v(x) + (y-x)\cdot p + \frac{\kappa}{2}|y-x|^2 \qquad y \in \mathbf{R}^d, \qquad (17)$$

where x is a point in $\mathbf{R}^d$, p is a vector of $\mathbf{R}^d$, and κ is a real number.

We are now ready to state the new continuous dependence result.

Theorem 3 (Second Continuous Dependence Result for Elliptic Equations). *Let u be the solution of the equation (4) and let v be any $\mathcal{C}^2(\mathbf{R}^d)$ function periodic in each coordinate with period 1. Then, for $\sigma \in \{-,+\}$, we have that*

$$|u - v|_\sigma \le \inf_{\bar\nu \ge 0, \epsilon > 0} \Phi_\sigma^{\bar\nu}(v;\epsilon), \qquad (18)$$

where

$$\Phi_\sigma^{\bar\nu}(v;\epsilon) = \sup_{(x,p)\in \mathcal{A}_\sigma(v;\epsilon)} \left(\sigma R_{\sigma\epsilon}^{\bar\nu}(v;x,p) + \frac{(\sqrt{\nu} - \sqrt{\bar\nu})^2}{\epsilon}\, d \right)^+ . \qquad (19)$$

The set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of elements (x,p) satisfying

$$(x,p) \in \mathbf{R}^d \times \mathbf{R}^d,$$
$$\sigma\{v(y) - P_v(x,p,\sigma/\epsilon\,;y)\} \le 0 \qquad \forall\, y \in \mathbf{R}^d.$$

Moreover, if $\bar\nu = 0$, the above estimate holds for $v \in \mathcal{C}^0(\mathbf{R}^d)$.

Some important remarks are in order:

a. The set $\mathcal{A}_\sigma(v;\epsilon)$ and the semi-differentials of v. By definition, the point (x,p) belongs to the set $\mathcal{A}_-(v;\epsilon)$ if the paraboloid $P_v(x,p,-1/\epsilon\,;\cdot)$ does not lie above the function v. This implies that for all $y \in \mathbf{R}^d$,

$$0 \le v(y) - P_v(x,p,-1/\epsilon\,;y) = v(y) - \left\{ v(x) + (y-x)\cdot p - \frac{1}{2\epsilon}|y-x|^2 \right\},$$

or, equivalently,

$$v(y) - \{v(x) + (y-x)\cdot p\} \geq -\frac{1}{2\epsilon}|\,y-x\,|^2.$$

This implies that p belongs to the *sub-differential* of v at x, $D^-v(x)$, which is defined to be the set of vectors $p \in \mathbf{R}^d$ such that

$$\lim_{y\to x}\,(v(y) - \{v(x) + (y-x)\cdot p\}) \geq 0. \tag{20}$$

Similar comments can be made in the case $\sigma = +$. Indeed, the point (x,p) belongs to the set $\mathcal{A}_+(v;\epsilon)$ if the paraboloid $P_v(x,p,1/\epsilon\,;\,\cdot)$ does not lie below the function v. In this case, p belongs to the *super-differential* of v at x, $D^+v(x)$, which is defined to be the set of vectors $p \in \mathbf{R}^d$ such that

$$\lim_{y\to x}\,(v(y) - \{v(x) + (y-x)\cdot p\}) \leq 0. \tag{21}$$

Note that, if v is smooth at x, the paraboloid $P_v(x,p,\sigma/\epsilon\,;\,\cdot)$ is *tangent* to v at x since in this case $D^+v(x) = D^-v(x) = \{\nabla v(x)\}$.

b. The dependence of the set $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ on ϵ. Note that if $\epsilon \leq \bar\epsilon$, then

$$\mathcal{A}_\sigma(v;\epsilon) \supset \mathcal{A}_\sigma(v;\bar\epsilon).$$

This means that the function $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ could decrease as ϵ increases. On the other hand, since

$$R_{\sigma\epsilon}^{\bar\nu}(v;x,p) = R_0^{\bar\nu}(v;x,p) + f(x) - f(x-\sigma\,\epsilon\,p) - \frac{\sigma\epsilon}{2}\,|\,p\,|^2,$$

the function $\Phi_\sigma^{\bar\nu}(v;\epsilon)$ can also increase as ϵ increases. Theorem 3 improves the result of Theorem 1 precisely because in many cases we do have that $\inf_{\epsilon>0}\Phi_\sigma^{\bar\nu}(v;\epsilon)$ is actually significantly smaller than $\lim_{\epsilon\downarrow 0}\Phi_\sigma^{\bar\nu}(v;\epsilon)$; see an analytical example in [2].

c. Recovering Theorem 1. From Theorem 3, we can deduce that, for $\bar\nu = \nu$,

$$|\,u-v\,|_\sigma \leq \lim_{\epsilon\downarrow 0}\Phi_\sigma^{\nu}(v;\epsilon) = \Phi_\sigma(v).$$

which means that we recover Theorem 1. To see this, we proceed as follows. First, since $\nu > 0$, note that $v \in \mathcal{C}^2(\mathbf{R}^d)$. In this case, it is not difficult to prove that

$$\lim_{\epsilon\downarrow 0}\mathcal{A}_\sigma(v;\epsilon) = \left\{(x,\nabla v(x)), x \in \mathbf{R}^d\right\},$$

and that, since $p = \nabla v(x)$,

$$\lim_{\epsilon\downarrow 0}\,(\sigma\,R_{\sigma\epsilon}^{\nu}(v;x,p))^+ = (\sigma\,R(v;x))^+\,.$$

Now, let us show that Theorem 3 *does not* break down when ν goes to zero. To do that, it is enough to show that we are able to use it to compare solutions of

(4) with different values of the parameter ν. That this is indeed the case is shown in the following result, where u_ν denotes the solution of (4).

Corollary 4 ($\{u_\nu\}_{\nu>0}$ is a Cauchy sequence). *We have*

$$\| u_\nu - u_{\bar\nu} \|_{L^\infty(\Omega)} \le \sqrt{2\,d}\,|\, f\,|_{W^{1,\infty}(\Omega)} \left| \sqrt{\nu} - \sqrt{\bar\nu} \right| .$$

Proof. Since for any $p \in \mathbf{R}^d$,

$$\begin{aligned} R^{\bar\nu}_{\sigma\epsilon}(u_{\bar\nu}; x, p) &= f(x) - f(x - \sigma\epsilon\, p) - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \epsilon\,|\,p\,|\,|\,f\,|_{W^{1,\infty}(\Omega)} - \frac{\epsilon}{2}\,|\,p\,|^2 \\ &\le \frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)}, \end{aligned}$$

we get, from Theorem 3,

$$\begin{aligned} |\,u_\nu - u_{\bar\nu}\,|_\sigma &\le \inf_{\epsilon\ge 0} \;\; \Phi^{\bar\nu}_\sigma(u_{\bar\nu};\epsilon) \\ &\le \inf_{\epsilon\ge 0} \;\; \left(\frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)} + \frac{(\sqrt{\nu}-\sqrt{\bar\nu})^2}{\epsilon}\,d \right) \\ &= \sqrt{2\,d}\,|\,f\,|_{W^{1,\infty}(\Omega)} \left| \sqrt{\nu} - \sqrt{\bar\nu} \right|, \end{aligned}$$

and the result follows from (5). This completes the proof. □

3.2 Proof of the Second Continuous Dependence Theorem for Elliptic Equations

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. We use the so-called *doubling-of-the-variables* technique which consists in considering the auxiliary function

$$\psi(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\epsilon},$$

instead of the auxiliary function $\psi(x, x) = u(x) - v(x)$ used previously.

We let $(\widehat{x}, \widehat{y}) \in \Omega \times \Omega$ be such that

$$\psi(\widehat{x}, \widehat{y}) \ge \psi(x, y) \qquad \forall x, y \in \Omega;$$

such a point exists since ψ is continuous and periodic on $\Omega \times \Omega$. Set $\widehat{p} = (\widehat{x} - \widehat{y})/\epsilon$.

We assume that $|\,u - v\,|_- > 0$, otherwise there is nothing to prove; also, we assume that $v \in \mathcal{C}^2(\mathbf{R}^d)$. In this case, we have

$$|\,u - v\,|_- = \sup_{x\in\Omega} \left\{ u(x) - v(x) \right\}$$

$$
\begin{aligned}
&= \sup_{x\in\Omega} \psi(x,x) \\
&\le \sup_{x,y\in\Omega} \psi(x,y) \\
&= \psi(\widehat{x},\widehat{y}) \\
&= u(\widehat{x}) - v(\widehat{y}) - \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon} \\
&= \quad \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\quad -\big[v(\widehat{y}) + H(\widehat{p}) - \bar{\nu}\Delta v(\widehat{y}) - f(\widehat{x}) + \frac{|\widehat{x}-\widehat{y}|^2}{2\epsilon}\,\big] \\
&\quad +\big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big] \\
&= \quad \big[u(\widehat{x}) + H(\widehat{p}) - \nu\Delta u(\widehat{x}) - f(\widehat{x})\big] \\
&\quad -\big[v(\widehat{y}) + H(\widehat{p}) - \bar{\nu}\Delta v(\widehat{y}) - f(\widehat{y}+\epsilon\, p) + \frac{|\,p\,|^2}{2\epsilon}\,\big] \\
&\quad +\big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big]
\end{aligned}
$$

by definition of $\widehat{p}$. Hence, by definition of the generalized residual $R_\epsilon^{\bar{\nu}}(\cdot;\cdot,\cdot)$, (16), we get

$$|\,u-v\,|_- \le R_0^{\nu}(u;\widehat{x},\widehat{p}) - R_{-\epsilon}^{\bar{\nu}}(v;\widehat{y},\widehat{p}) + \big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big].$$

Since the mapping $x \mapsto \psi(x,\widehat{y})$ has a maximum at $x=\widehat{x}$, we have $0 = \nabla_x\psi(\widehat{x},\widehat{y})$ and so

$$\widehat{p} = \nabla u(\widehat{x}).$$

Now, since u is the exact solution of equation (4), $R_0^{\nu}(u;\widehat{x},\widehat{p}) = 0$, and hence,

$$|\,u-v\,|_- \le -R_{-\epsilon}^{\bar{\nu}}(v;\widehat{y},\widehat{p}) + \big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big].$$

To deal with the last term of the right-hand side, we use the fact that the Hessian of ψ at $(\widehat{x},\widehat{y})$, $\mathcal{H}\psi(\widehat{x},\widehat{y})$, is negative semi-definite. Note that we have

$$\mathcal{H}\psi(\widehat{x},\widehat{y}) = \begin{pmatrix} \mathcal{H}u(\widehat{x}) - \frac{1}{\epsilon}\,Id & \frac{1}{\epsilon}\,Id \\ \frac{1}{\epsilon}\,Id & -\mathcal{H}v(\widehat{y}) - \frac{1}{\epsilon}\,Id \end{pmatrix},$$

where Id is the identity matrix, and $\mathcal{H}u$ and $\mathcal{H}v$ are the Hessians of u and v, respectively.

Now, let us denote by e_i the d-dimensional vector whose j-th entry is δ_{ij} and set $\eta_i^t = (\sqrt{\nu}e_i^t, \sqrt{\bar{\nu}}e_i^t)$. Since $\eta_i^t\ \mathcal{H}\psi(\widehat{x},\widehat{y})\ \eta_i \le 0$, we have that

$$\sum_{i=1}^{d} \eta_i^t\ \mathcal{H}\psi(\widehat{x},\widehat{y})\ \eta_i \le 0,$$

which reads

$$\big[\nu\Delta u(\widehat{x}) - \bar{\nu}\Delta v(\widehat{y})\big] \le \frac{(\sqrt{\nu}-\sqrt{\bar{\nu}})^2}{\epsilon}\, d.$$

At this point of the proof, it is important to stress the fact that obtaining this elegant inequality is a **crucial** *step for proving the theorem and that it is the doubling-of-the-variables technique that allows this to happen!*

We can now write

$$| u - v |_- \leq -R^{\bar\nu}_{-\epsilon}(v; \widehat{y}, \widehat{p}) + \frac{(\sqrt{\nu} - \sqrt{\bar\nu})^2}{\epsilon}\, d.$$

To prove the estimate, it only remains to show that $(\widehat{y}, \widehat{p})$ belongs to $\mathcal{A}_\sigma(v;\epsilon)$. To do that, we note that since $\psi(\widehat{x}, \widehat{y}) \geq \psi(\widehat{x}, y)$ for all $y \in \Omega$, we have

$$v(y) \geq v(\widehat{y}) + \frac{|\widehat{x} - \widehat{y}|^2}{2\epsilon} - \frac{|\widehat{x} - y|^2}{2\epsilon} = v(\widehat{y}) + \widehat{p}\cdot(y - \widehat{y}) - \frac{|y - \widehat{y}|^2}{2\epsilon},$$

which is nothing but the so-called paraboloid test:

$$v(y) - Pv(\widehat{y}, \widehat{p}, -1/\epsilon; y) \leq 0 \qquad \forall\, y \in \mathbf{R}^d.$$

Here, it is also important to stress the fact that it is the doubling-of-the-variables technique that allows us to obtain the paraboloid test, which is a **very** *important feature of the error estimate!*

This completes the proof of Theorem 3 for $v \in \mathcal{C}^2(\mathbf{R}^d)$.

Now, in the case $\bar\nu = 0$, it is clear that v does not need to be in $\mathcal{C}^0(\mathbf{R}^d)$; we only need to consider $v \in \mathcal{C}^0(\mathbf{R}^d)$. Indeed, it is not difficult to verify that the above proof is valid also in this case provided we modify the argument about the Hessian of ψ as follows: Since the mapping $x \mapsto \psi(x, \widehat{y})$ has a maximum at $x = \widehat{x}$, we have that

$$\Delta_x \psi(\widehat{x}, \widehat{y}) \leq 0,$$

that is, that

$$\Delta u(\widehat{x}) \leq d/\epsilon.$$

This completes the proof of Theorem 3 for $\bar\nu = 0$ and $v \in \mathcal{C}^0(\mathbf{R}^d)$.

4 Continuous Dependence Results for Hamilton-Jacobi Equations

Corollary 4 shows that there exists a unique limit of the sequence $\{ u_\nu \}_{\nu > 0}$ which also belongs to $\mathcal{C}^0(\mathbf{R}^d)$ and is periodic in each coordinate with period 1. This limit is the so-called *viscosity solution* of the Hamilton-Jacobi equation

$$u + H(\nabla u) = f \qquad \text{in } \mathbf{R}^d. \tag{22}$$

4.1 Preservation of continuous dependence under the limit $\nu \downarrow 0$

The following result is a direct consequence of Theorem 3 and Corollary 4. This shows how the continuous dependence result contained in Theorem 3 is *preserved* when we take the limit $\nu \downarrow 0$. In what follows, we write $R_{\sigma\epsilon}$ instead of $R^0_{\sigma\epsilon}$

Theorem 5 (Continuous Dependence Result for Hamilton-Jacobi equations). *Let u be the viscosity solution of* (22) *and let v be any $\mathcal{C}^0(\mathbf{R}^d)$ function periodic in each coordinate with period* 1. *Then, for $\sigma \in \{-, +\}$, we have that*

$$| u - v |_\sigma \leq \inf_{\epsilon > 0} \Phi_\sigma(v; \epsilon), \tag{23}$$

where

$$\Phi_\sigma(v;\epsilon) = \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}(v;x,p)\right)^+ . \tag{24}$$

The set $\mathcal{A}_\sigma(v;\epsilon)$ *is the set of elements* (x,p) *satisfying*

$$\begin{aligned} &(x,p) \in \mathbf{R}^d \times \mathbf{R}^d, \\ &\sigma\,\{v(y) - P_v(x,p,\sigma/\epsilon\,;\,y)\} \le 0 \qquad \forall y \in \mathbf{R}^d. \end{aligned}$$

Let us point out that *all* the properties described in subsection 2.1 for the exact solution of the elliptic equation (4) hold for the viscosity solution of the Hamilton-Jacobi equation (22). They can be easily deduced from the above continuous dependence result.

4.2 Characterization of the viscosity solution

Moreover, from Theorem 5 we can obtain the following characterization of the viscosity solutions.

Corollary 6 (Characterization of the viscosity solution). *The viscosity solution of the Hamilton-Jacobi equation* (22) *is the only function* u *in* $\mathcal{C}^0(\mathbf{R}^d)$*, periodic in each coordinate with period* 1*, such that for all* x *in* $\mathbf{R}^d$*,*

$$u(x) + H(p) - f(x) \le 0 \quad \forall\, p \in D^+u(x),$$

and

$$u(x) + H(p) - f(x) \ge 0 \quad \forall\, p \in D^-u(x).$$

Proof. Let u be the viscosity solution of the Hamilton-Jacobi equation (22) and let v be a function such that, for all x in $\mathbf{R}^d$,

$$v(x) + H(p) - f(x) \ge 0 \quad \forall\, p \in D^-v(x).$$

Then, from Theorem 5 we get

$$\begin{aligned} |\,u - v\,|_- &\le \inf_{\epsilon>0} \Phi_-(v;\epsilon) \\ &= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(-R_{-\epsilon}(v;x,p)\right)^+ \\ &= \inf_{\epsilon>0} \sup_{(x,p)\in\mathcal{A}_\sigma(v;\epsilon)} \left(f(x+\epsilon\, p) - f(x)\right)^+ \\ &\le \inf_{\epsilon>0} |\,f\,|_{W^{1,\infty}(\Omega)}\,|\,v\,|_{W^{1,\infty}(\Omega)}\,\epsilon \\ &= 0. \end{aligned}$$

Similarly, if v is such that

$$v(x) + H(p) - f(x) \ge 0 \quad \forall\, p \in D^-v(x),$$

we conclude in a similar manner that

$$|\, u - v \,|_+ \leq 0.$$

This implies that $u \equiv v$ and the Corollary is proved. □

5 Continuous Dependence Results for Monotone Schemes

5.1 Monotone schemes

The numerical schemes we are going to consider determine the values of a function u_h on a grid G_h which is, of course, periodic with period 1 in each of the canonical directions of $\mathbf{R}^d$. These schemes are of the form

$$u_h(y) + \widehat{H}_y(\partial_{\delta_y} u_h(y)) = f(y) \qquad \forall\, y \in G_h, \tag{25}$$

where $\widehat{H}_y(\partial_{\delta_y} u_h(y))$ is a discrete version of $H(\nabla u(y))$ and

$$\partial_{\delta_y} u_h(y) = (\partial_{\delta_{1,y}} u_h(y), \ldots, \partial_{\delta_{N_y,y}} u_h(y)),$$

where

$$\partial_{\delta_{i,y}} u_h(y) = \frac{u_h(y) - u_h(y - \delta_{i,y})}{|\, \delta_{i,y} \,|} \text{ where } y - \delta_{i,y} \in G_h \qquad i = 1, \ldots, N_y.$$

The numerical Hamiltonian $\widehat{H}$ has the following properties:

(i) Consistency: $\widehat{H}_y(\partial_{\delta_y} u_h(y)) = H(p)$ if $\nabla u_h = p \in \mathbf{R}^d$.

(ii) Monotonicity: $\widehat{H}_y$ is non-decreasing in each of its arguments.

(iii) Global smoothness: $|\, \widehat{H}_y(z_1) - \widehat{H}_y(z_2) \,| \leq L \,\|\, z_1 - z_2 \,\|_{\ell^\infty}$.

The first property ensures that we are approximating the viscosity solution with the correct Hamilton-Jacobi equation. The second property is the *key* property on the numerical Hamiltonian that will allow us to obtain the continuous dependence result we seek. The third property, is not really necessary but does simplify the proofs.

The existence of at least one solutions u_h can be easily proved by using the classical Leray-Schauder fixed point theorem; see the Appendix. The uniqueness will follow from the continuous dependence result we state in the next subsection.

The classical example of a monotone scheme for Hamilton-Jacobi equation is the so-called Lax-Friedrichs scheme. The Lax-Friedrichs scheme, see [8, 16, 2], on the uniform Cartesian grid

$$G_h = \{ (x_0 + (i-1)\Delta x, y_0 + (j-1)\Delta y) \}$$

reads as follows:

$$v_{i,j} + H\left(\frac{v_{i+1,j}-v_{i-1,j}}{2\Delta x}, \frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right)$$
$$-\omega_x \frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\Delta x^2} - \omega_y \frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\Delta y^2} = f_{i,j},$$

where

$$\omega_x = \sup_{(x,y)\in\mathbf{R}^d} \frac{1}{2}\left| H_1\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)\right| \Delta x,$$
$$\omega_y = \sup_{(x,y)\in\mathbf{R}^d} \frac{1}{2}\left| H_2\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)\right| \Delta y,$$

and $H_i(p_1,p_2) = \frac{\partial H}{\partial p_i}(p_1,p_2)$ for $i=1,2$.

Since the grid G_h is Cartesian, for each $y=(i,j)\in G_h$ we have $N_y = 4$; the quantities $\partial_{\delta_i,y} v(y)$ are thus the following

$$\partial_{-\Delta x} v_{i,j} = \frac{v_{i,j}-v_{i+1,j}}{\Delta x}, \qquad \partial_{\Delta x} v_{i,j} = \frac{v_{i,j}-v_{i-1,j}}{\Delta x},$$
$$\partial_{-\Delta y} v_{i,j} = \frac{v_{i,j}-v_{i,j+1}}{\Delta y}, \qquad \partial_{\Delta y} v_{i,j} = \frac{v_{i,j}-v_{i,j-1}}{\Delta y};$$

therefore,

$$\widehat{H} = H\left(\frac{1}{2}(\partial_{\Delta x} v_{i,j} - \partial_{-\Delta x} v_{i,j}), \frac{1}{2}(\partial_{\Delta y} v_{i,j} - \partial_{-\Delta y} v_{i,j})\right)$$
$$+\frac{\omega_x}{\Delta x}(\partial_{-\Delta x} v_{i,j} + \partial_{\Delta x} v_{i,j}) + \frac{\omega_y}{\Delta y}(\partial_{-\Delta y} v_{i,j} + \partial_{\Delta y} v_{i,j}).$$

It is easy to verify that the above $\widehat{H}$ satisfies properties (i), (ii) and (iii).

Other examples of schemes satisfying these three properties are the monotone schemes of Crandall and Lions [8], the intrinsic monotone scheme of Abgrall [1] and the monotone schemes of Kossioris, Makridakis and Souganidis [12].

5.2 A Continuous Dependence Result for Monotone schemes

The main result of this section gives upper bounds for discrete versions of the semi-norms used for the continuous case:

$$|\, u_h - v\,|_{-,G_h} = \sup_{x\in G_h} (\, u_h(x) - v(x)\,)^+, \qquad |\, u_h - v\,|_{+,G_h} = \sup_{x\in G_h} (\, v(x) - u_h(x)\,)^+,$$

where $\eta^+ \equiv \max\{0,\eta\}$. Note that an upper bound on the L^∞-norm of $u_h - v$

$$\|\, u_h - v\,\|_{L^\infty(G_h)} = \max\{|\, u_h - v\,|_{+,G_h}, |\, u_h - v\,|_{-,G_h}\}.$$

These upper bounds are given in terms of what could be considered to be a generalization of the classical truncation error. Next, we define this quantity.

We begin by introducing the expression

$$\mathcal{D}_{\delta,y;\epsilon,p}v(\widehat{y}) = \frac{v(\widehat{y}) - v(y)}{|\,\delta\,|} + \frac{\Theta_\epsilon(y - \widehat{y} + \delta, p)}{|\,\delta\,|},$$

where

$$\Theta_\epsilon(z,p) = p\cdot z + \frac{1}{2\epsilon}|\,z\,|^2.$$

By taking the particular choices $y = \widehat{y} - \delta$ and $y = \widehat{y}$, the above expression will be used to approximate directional derivatives, super- or sub-differentials, or finite differences of the function v.

Indeed, for the first choice, we have

$$\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p}v(\widehat{y}) = \frac{v(\widehat{y}) - v(\widehat{y}-\delta)}{|\,\delta\,|} + \frac{\Theta_\epsilon(0,p)}{|\,\delta\,|} = \partial_\delta v(\widehat{y}). \tag{26}$$

In other words, for the special choice of the auxiliary parameter $y = \widehat{y} - \delta$, the quantity under consideration is nothing but a finite difference of u in the δ-direction. Of course, if u is a smooth function, we have that

$$\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p}v(\widehat{y}) = \nabla v(\widehat{y})\cdot\frac{\delta}{|\,\delta\,|} + \mathcal{O}(|\,\delta\,|),$$

for small $|\,\delta\,|$; that is, $\mathcal{D}_{\delta,\hat{y}-\delta;\epsilon,p}v(\widehat{y})$ approximates the partial derivative of v at $\widehat{y}$ in the δ-direction.

For the second choice, we have

$$\mathcal{D}_{\delta,\hat{y};\epsilon,p}v(\widehat{y}) = \frac{v(\widehat{y}) - v(\widehat{y})}{|\,\delta\,|} + \frac{\Theta_\epsilon(\delta,p)}{|\,\delta\,|} = p\cdot\frac{\delta}{|\,\delta\,|} + \frac{|\,\delta\,|}{2\epsilon}, \tag{27}$$

and so, if $|\,\delta\,|$ is small,

$$\mathcal{D}_{\delta,\hat{y};\epsilon,p}v(\widehat{y}) = p\cdot\frac{\delta}{|\,\delta\,|} + \mathcal{O}(|\,\delta\,|).$$

Since in our applications, p will belong to super- or sub-differentials of viscosity solutions, the quantity $\mathcal{D}_{\delta,\hat{y};\epsilon,p}v(\widehat{y})$ provides an approximation to the component of p in the δ-direction.

Now, we introduce the *generalized truncation error* of v at the point $y \in \mathbf{R}^d$:

$$T_{\mathbf{y};\epsilon}(v;y,p) = v(y) + \widehat{H}_{y-\epsilon\,p}(\mathcal{D}_{\delta_{y-\epsilon\,p},\mathbf{y};\epsilon,p}v(y)) - f(y-\epsilon\,p) - \frac{|\,p\,|^2}{2}\epsilon, \tag{28}$$

where $\mathbf{y} = (y_1,\ldots,y_{N_y})$ and

$$\mathcal{D}_{\delta_y,\mathbf{y};\epsilon,p}v(y) = (\mathcal{D}_{\delta_1,y_1;\epsilon,p}v(y),\ldots,\mathcal{D}_{\delta_{N_y},y_{N_y};\epsilon,p}v(y)).$$

Note that for the above expression to have a meaning, the point $x = y - \epsilon\,p$ must belong to the grid G_h. Note also that if we set $y_i = y - \delta_i$ for $i = 1,\ldots,N_y$ and then set $\epsilon = 0$, we get,

$$T_{\mathbf{y};0}(u;y,p) = u(y) + \widehat{H}_y(\partial_{\delta_y}u(y)) - f(y),$$

by using (26); since the above expression is nothing but the classical truncation error of v at $y \in G_h$, our terminology is justified.

We are now ready to state our main result.

Theorem 7 (Continuous Dependence Result for Monotone Schemes). *Let u_h be the approximate solution given by the monotone scheme (25) and let v be an arbitrary continuous function periodic of period 1 in each of the canonical directions of $\mathbf{R}^d$. Then, for $\sigma \in \{-,+\}$, we have that*

$$| u_h - v |_{\sigma,G_h} \leq \inf_{\epsilon>0} \Psi_\sigma(v;\epsilon), \tag{29}$$

where

$$\Psi_\sigma(v;\epsilon) = \sup_{(y,p)\in\mathcal{A}_{h,\sigma}(v;\epsilon)} \inf_{\mathbf{y}\in\mathbf{R}^{N_y\times d}} \left(\sigma\, T_{\mathbf{y};\sigma\epsilon}(v;y,p)\right)^+. \tag{30}$$

The set $\mathcal{A}_{h,\sigma}(v;\epsilon)$ is the set of elements (y,p) satisfying

$$\begin{aligned}&(y-\sigma\,\epsilon\, p, p) \in G_h \times \mathbf{R}^d,\\ &\sigma\,\{v(\zeta) - P_v(y,p,\sigma/\epsilon\,;\,\zeta)\} \leq 0 \qquad \forall\, \zeta \in \mathbf{R}^d.\end{aligned}$$

It is clear that this result can be considered to be a discrete version of Theorem 5 for the Hamilton-Jacobi equations. In this sense, we could say, in a very loose way, that the continuous dependence result of Theorem 5 has been preserved under discretization of the Hamilton-Jacobi equations by monotone schemes.

a. The a priori error estimate. Note that, if v is taken to be the viscosity solution u of the Hamilton-Jacobi equations (22), Theorem 7 allows us to estimate the L^∞-error between u_h and u solely in terms of u. Since this can be done only *before* the computation of u_h, estimates like this are called *a priori error estimates*. Their usefulness resides in the fact that they tell us the accuracy you can expect from the approximation.

In our case, we have the following result.

Corollary 8 (A Priori Error Estimates). *Let u be the viscosity solution of the equation (22) and let v be the approximate solution given by the monotone scheme (25). Then, if $f \in W^{1,\infty}(\Omega)$,*

$$\| u - v \|_{L^\infty(G_h)} \leq L^{\frac{1}{2}}\, | f |_{W^{1,\infty}(\Omega)}\, h^{\frac{1}{2}},$$

where $h = \max_{y\in G_h} \max_{i=1,\dots,N_y} |\,\delta_i\,|$ and L is the Lipschitz constant of the numerical Hamiltonian. Moreover, if $u \in W^{2,\infty}(\Omega)$,

$$\| u - v \|_{L^\infty(G_h)} \leq \frac{d}{2}\, L\, | u |_{W^{2,\infty}(\Omega)}\; h.$$

The proof of this result will be given in subsection 5.3.

b. Uniqueness of the approximation u_h. We can prove the uniqueness of the approximate solution u_h by a simple application of Theorem 7. We proceed by contradiction. Assume that there are two different solutions v_1 and v_2. Setting $u_h = v_1$ and $v = v_2$ in Theorem 7, we get,

$$|\, v_1 - v_2 \,|_{\sigma,G_h} \leq \Psi(v_1;0) = \sup_{y\in G_h} (-\sigma\,(v_1(y) + \widehat{H}_y(\partial_{\delta_y} v_1(y)) - f(y)))^+ = 0.$$

This implies $v_1 = v_2$ and the uniqueness of u_h follows.

c. A maximum principle. If in Theorem 7 we set $v = 0$ and $\epsilon = 0$, we get that

$$-\sup_{\zeta\in G_h} (H(0) - f(\zeta))^+ \leq v(y) \leq \sup_{\zeta\in G_h} (f(\zeta) - H(0))^+ \qquad \forall\, y \in G_h.$$

d. The $\mathbf{L}^\infty$-contraction property Let us call $u_{h,f}$ the solution of (25) with right-hand side f. Then, setting $u_h = u_{h,f}$, $v = u_{h,g}$ and $\epsilon = 0$ in Theorem 7, we get

$$|\, u_{h,f} - u_{h,g} \,|_{\sigma,G_h} \leq |\, f - g \,|_{\sigma,G_h}.$$

The above result immediately implies the inequality

$$\|\, u_{h,f} - u_{h,g} \,\|_{L^\infty(G_h)} \leq \|\, f - g \,\|_{L^\infty(G_h)},$$

which is nothing but the discrete version of the so-called L^∞-contraction property for viscosity solutions.

5.3 Proof of the A Priori Error Estimates

a. The case $u \in W^{2,\infty}(\Omega)$.

Taking $y_i = y - \delta_i$ for $i = 1, \ldots, N_y$ and then $\epsilon = 0$, we obtain, by the definition of the generalized truncation error (28),

$$\Theta = T_{\mathbf{y};0}(u;y,p) = u(y) + \widehat{H}_y(\partial_{\delta_y} u(y)) - f(y) = \widehat{H}_y(\partial_{\delta_y} u(y)) - H(\nabla u(y)),$$

since u is the smooth viscosity solution. Next, we set $\mathcal{L}u(\zeta) = u(y) + \nabla u(y)\cdot(\zeta - y)$, and invoke the consistency property of the numerical Hamiltonian to get

$$\Theta = \widehat{H}_y(\partial_{\delta_y} u(y)) - \widehat{H}_y(\partial_{\delta_y} \mathcal{L}u(y)).$$

Finally, by using the smoothness of $\widehat{H}$ and a simple Taylor expansion, we finally obtain,

$$\begin{aligned}
|\,\Theta\,| &\leq L\ \left\|\, \partial_{\delta_y} u(y) - \partial_{\delta_y} \mathcal{L}u(y) \,\right\|_{\ell^\infty} \\
&= L \max_{i=1,\ldots,N_y} \left|\, \frac{u(y) - u(y-\delta_i)}{|\,\delta_i\,|} - \nabla u(y)\cdot \frac{\delta_i}{|\,\delta_i\,|} \,\right| \\
&\leq \frac{d}{2}\, L\, |\, u \,|_{W^{2,\infty}(\Omega)}\, h.
\end{aligned}$$

This completes the proof of Corollary 8 in the case of $u \in W^{2,\infty}(\Omega)$.

b. The case $f \in W^{1,\infty}(\Omega)$. For each $(y,p) \in \mathcal{A}_{h,\sigma}(u,\epsilon)$ and $\sigma \in \{+,-\}$, consider the quantity

$$\Theta = \sigma\, T_{\mathbf{y};\sigma\epsilon}(u;y,p) = \sigma\left(u(y) + \widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y)) - f(y-\sigma\epsilon\, p)\right) - \frac{1}{2}\epsilon|\,p\,|^2,$$

by the definition (28). Since $p \in D^\sigma u(y)$, we have

$$\sigma(u(y) + H(p) - f(y)) \leq 0,$$

by definition of the viscosity solution. This implies that

$$\begin{aligned}\Theta &\leq \sigma\left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y)) - H(p) - f(y-\sigma\epsilon\, p) + f(y)\right) - \frac{1}{2}\epsilon|\,p\,|^2 \\ &\leq \sigma\left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y)) - H(p)\right) + \epsilon|\,f\,|_{W^{1,\infty}(\Omega)}\,|\,p\,| - \frac{1}{2}\epsilon|\,p\,|^2 \\ &\leq \sigma\left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y)) - H(p)\right) + \frac{\epsilon}{2}|\,f\,|^2_{W^{2,\infty}(\Omega)}.\end{aligned}$$

Now, if we set $\mathcal{L}u(\zeta) = u(y) + p\cdot(\zeta - y)$, we have

$$H(p) = \widehat{H}_{y-\sigma\epsilon\, p}(\partial_{\delta_{y-\sigma\epsilon\, p}}\mathcal{L}u(y)),$$

by the consistency of the numerical Hamiltonian, and so,

$$\begin{aligned}\Theta &\leq \sigma\left(\widehat{H}_y(\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y)) - \widehat{H}_{y-\sigma\epsilon\, p}(\partial_{\delta_{y-\sigma\epsilon\, p}}\mathcal{L}u(y))\right) + \frac{\epsilon}{2}|\,f\,|^2_{W^{1,\infty}(\Omega)} \\ &\leq L\,\|\,\mathcal{D}_{\delta_{y-\sigma\epsilon\, p},\mathbf{y};\sigma\epsilon,p}u(y) - \partial_{\delta_{y-\sigma\epsilon\, p}}\mathcal{L}u(y)\,\|_{\ell^\infty} + \frac{\epsilon}{2}|\,f\,|^2_{W^{1,\infty}(\Omega)},\end{aligned}$$

by the smoothness property of $\widehat{H}$. Finally, since

$$\partial_{\delta_i}\mathcal{L}u(y) = p\cdot\frac{\delta_i}{|\,\delta_i\,|} \qquad \text{and} \qquad \mathcal{D}_{\delta_i,\widehat{x};\sigma\epsilon,p}u(y) = p\cdot\frac{\delta_i}{|\,\delta_i\,|} + \frac{|\,\delta_i\,|}{2\sigma\epsilon},$$

by (27), we obtain by taking $y_i = y$,

$$\Theta \leq L\,\frac{h}{2\epsilon} + \frac{\epsilon}{2}\,|\,f\,|^2_{W^{1,\infty}(\Omega)}.$$

The result follows by minimizing the right-hand side with respect to ϵ. This completes the proof of Corollary 8 in the case of $f \in W^{1,\infty}(\Omega)$.

5.4 Proof of the Continuous Dependence Result for Monotone Schemes

Note how the proof of this result is a discrete version of the proof of the second continuous dependence theorem for elliptic equations.

We prove the result for $\sigma = -$; the proof for the case $\sigma = +$ is similar. We assume that $|\,u_h - v\,|_{-,G_h} > 0$, otherwise there is nothing to prove. Given $\epsilon > 0$, we define the auxiliary function

$$\psi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\epsilon},$$

and let $(\widehat{x}, \widehat{y}) \in \mathbf{R}^d \times G_h$ be such that

$$\psi(\widehat{x}, \widehat{y}) \geq \psi(x, y) \qquad \forall\ (x, y) \in G_h \times \mathbf{R}^d.$$

The existence of such a point easily follows from the fact that both u_h and v are continuous and periodic with the same period. Set $\widehat{p} = (\widehat{x} - \widehat{y})/\epsilon$.

So, since $|\, u_h - v\,|_{-,G_h} > 0$, we have

$$\begin{aligned}
|\, u_h - v\,|_{-,G_h} &= \sup_{x \in G_h} \{u_h(x) - v(x)\} \\
&= \sup_{x \in G_h} \psi(x, x) \\
&\leq \sup_{(x,y) \in G_h \times \mathbf{R}^d} \psi(x, y) \\
&= \psi(\widehat{x}, \widehat{y}) \\
&= u_h(\widehat{x}) - v(\widehat{y}) - \frac{|\widehat{x} - \widehat{y}|^2}{2\epsilon} \\
&= \quad \big[u_h(\widehat{x}) + \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x})) - f(\widehat{x})\big] \\
&\quad - \big[v(\widehat{y}) + \widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}}, \mathbf{y}; -\epsilon, \hat{p}} v(\widehat{y})) + f(\widehat{x}) - \frac{|\widehat{x} - \widehat{y}|^2}{2\epsilon}\big] \\
&\quad + \big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}}, \mathbf{y}; -\epsilon, \hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big] \\
&= -\big[v(\widehat{y}) + \widehat{H}_{\hat{y} + \epsilon\,\hat{p}}(\mathcal{D}_{\delta_{\hat{y}+\epsilon\,\hat{p}}, \mathbf{y}; -\epsilon, \hat{p}}\ v(\widehat{y})) - f(\widehat{y} + \epsilon\,\widehat{p}) + \frac{|\,\widehat{p}\,|^2}{2}\epsilon\big] \\
&\quad + \big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}}, \mathbf{y}; -\epsilon, \hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big]
\end{aligned}$$

by the definition of the approximate solution u_h, (25), and the definition of $\widehat{p}$. Hence, by the definition of the generalized truncation error, (28), we get

$$|\, u_h - v\,|_{-,G_h} \leq -T_{\mathbf{y}; -\epsilon}(v; \widehat{y}, \widehat{p}) + \big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}}, \mathbf{y}; -\epsilon, \hat{p}}\ v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big].$$

Next, we show that the last term in the above inequality is non-positive. We proceed as follows. First we show that

$$\partial_{\delta_{\hat{x}}} u_h(\widehat{x}) \geq \mathcal{D}_{\delta_{\hat{x}}, \mathbf{y}; -\epsilon, \hat{p}}\ v(\widehat{y}).$$

We begin by noting that the inequality

$$\psi(\widehat{x}, \widehat{y}) \geq \psi(\widehat{x} - \delta, y) \qquad \forall\ (\widehat{x} - \delta, y) \in G_h \times \mathbf{R}^d,$$

can be rewritten as follows:

$$u_h(\widehat{x}) - u_h(\widehat{x} - \delta) \geq v(\widehat{y}) - v(y) + \Theta, \tag{31}$$

where

$$\Theta = \frac{|\widehat{x} - \widehat{y}|^2}{2\epsilon} - \frac{|y - \widehat{x} + \delta|^2}{2\epsilon} = \widehat{p} \cdot (y - \widehat{y} + \delta) - \frac{1}{2\epsilon}|y - \widehat{y} + \delta\,|^2 = \Theta_{-\epsilon}(y - \widehat{y} + \delta, \widehat{p}).$$

Since this implies that

$$\partial_\delta u_h(\widehat{x}) = \frac{u_h(\widehat{x}) - u_h(\widehat{x}-\delta)}{|\,\delta\,|} \le \frac{v(\widehat{y}) - v(y)}{|\,\delta\,|} + \frac{\Theta_{-\epsilon}(y-\widehat{y}+\delta,\widehat{p})}{|\,\delta\,|} = \mathcal{D}_{\delta,y;-\epsilon,\hat{p}}\, v(\widehat{y}),$$

we get, *by using in a* **crucial** *way the monotonicity of the numerical Hamiltonian* $\widehat{H}$, that

$$\big[\widehat{H}_{\hat{x}}(\mathcal{D}_{\delta_{\hat{x}},\mathbf{y};-\epsilon,\hat{p}} v(\widehat{y})) - \widehat{H}_{\hat{x}}(\partial_{\delta_{\hat{x}}} u_h(\widehat{x}))\big] \le 0.$$

This implies

$$|\,u_h - v\,|_{-,G_h} \le -T_{\mathbf{y};-\epsilon}(v;\widehat{y},\widehat{p}).$$

The estimate follows if we show that $(\widehat{y},\widehat{p})$ belongs to $\mathcal{A}_{h,-}(v;\epsilon)$. By definition of $\widehat{p}$, we have that $\widehat{y}+\epsilon\,\widehat{p} = \widehat{x} \in G_h$. Now, set $\delta = 0$ in (31) to get

$$0 \ge v(\widehat{y}) - v(y) + \widehat{p}\cdot(y-\widehat{y}) - \frac{1}{2\epsilon}|\,y-\widehat{y}\,|^2,$$

or, equivalently,

$$v(y) - P_v(\widehat{y},\widehat{p},-1/\epsilon;y) \ge 0 \qquad \forall\, y \in \mathbf{R}^d.$$

This completes the proof of Theorem 7. □

6 Some extensions

Extensions of the continuous dependence results contained in this paper to the case of more complicated Hamiltonians can be easily done by following, for example, Crandall and Lions [6]. In that paper, an extension to the time-dependent case is also carried out. For continuous dependence results for parabolic equations, see the results of Cockburn, Gripenberg and Londen[5].

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Appendix: Existence of the approximate solution given by a monotone scheme

Proposition 9 (Existence of u_h). *There exists a solution u_h given by the monotone scheme* (25).

Proof. We establish the existence by using the classical Leray-Schauder fixed-point theorem (see, for example, page 162 in [14]).

First, we enumerate the points y_i of the grid G_h and identify the function v defined on G_h to the point $(v_1, \ldots, v_N)$, where $v_i = v(y_i)$. Then, we consider the mapping $\mathcal{F} : \mathbf{R}^N \to \mathbf{R}^N$ defined by

$$(\mathcal{F}(v))(y) = -\widehat{H}_y(\partial_{\delta_y} v(y)) + f(y) \qquad \forall\, y \in G_h,$$

since a fixed point of this mapping is a solution of the numerical scheme (25) and vice versa. Choosing $r = \| H(0) - f \|_{l^\infty(\mathbf{R}^N)}$, the Leray-Schauder fixed-point theorem guarantees the existence of a fixed point of $\mathcal{F}$ in the ball $\overline{B}(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} \leq r\}$ if $\mathcal{F}$ is a continuous mapping from $B(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} < r\} \subset \mathbf{R}^N$ to $\mathbf{R}^N$ and $\mathcal{F}(v) \neq \lambda v$ whenever $\lambda > 1$ and $v \in \partial B(r) = \{v \in \mathbf{R}^N : \| v \|_{\ell^\infty} = r\}$.

Given that we have assumed the numerical Hamiltonian to be locally Lipschitz, it is clear that $\mathcal{F}$ is a continuous mapping. Assuming $v \in \partial B(r)$, i.e., $\| v \|_{\ell^\infty} = r$, there must exist at least one index $i_0 (1 \leq i_0 \leq N)$ such that $v_{i_0} = v(y_{i_0}) = r$ or $v_{i_0} = v(y_{i_0}) = -r$. If $v_{i_0} = r$, then $v_{i_0} \geq v_i$ for $1 \leq i \leq N$; therefore, $\frac{v_{i_0} - v_i}{|\delta_{i_0,i}|} \geq 0$ for $1 \leq i \leq N$. Furthermore, **by the monotonicity** of $\widehat{H}_y$, we have that $\widehat{H}_{y_{i_0}}(\partial_{\delta_{y_{i_0}}} v(y_{i_0})) \geq \widehat{H}_{y_{i_0}}(0) = H(0)$. Hence,

$$-\widehat{H}_{y_{i_0}}(\partial_{\delta_{y_{i_0}}} v(y_{i_0})) + f(y_{i_0}) \leq -H(0) + f(y_{i_0}) \leq r < \lambda r = \lambda v_{i_0};$$

that is, $(\mathcal{F}(v))(y_{i_0}) \neq \lambda v(y_{i_0})$. Similarly, if $v_{i_0} = -r$, we still have $(\mathcal{F}(v))(y_{i_0}) \neq \lambda v(y_{i_0})$. Thus $(\mathcal{F}(v)) \neq \lambda v$ for $v \in \partial B(r)$. All in all, the assumptions of the Leray-Schauder theorem hold, hence we conclude that there exists a solution $v = u_h$ given by the monotone scheme (25) in the ball $\overline{B}(r)$. $\square$

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