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A Paraxial Formulation for the Viscosity Solution of Quasi-P Eikonal Equations

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Abstract—Stationary quasi-P eikonal equations, stationary Hamilton-Jacobi equations, arise from the asymptotic approximation of anisotropic wave propagation. A paraxial formulation of the quasi-P eikonal equation results in a paraxial quasi-P eikonal equation, an evolution Hamilton-Jacobi equation in a preferred direction, which provides a fast and efficient way for computing viscosity solutions of quasi-P eikonal equations. Under the assumption that the initial condition is continuous (and possibly unbounded), the (unbounded) viscosity solution exists and is unique for the paraxial quasi-P eikonal equations.

Keywords—Paraxial eikonal equations, Hamilton-Jacobi equations, Viscosity solutions, First-arrival traveltimes.

1. INTRODUCTION

The high frequency asymptotic theory originated in the field of geometrical optics [1], and the progressing wave expansion for the asymptotic method involves a "phase function" and an infinite sequence of "amplitude functions". The phase function satisfies a nonlinear partial differential equation of first order called the eikonal equation; the amplitude functions satisfy a sequence of successive "transport equations" [2]. The asymptotic method has been successfully used to construct the asymptotic solution for acoustic wave equations [2] and linear elastodynamic wave equations [3–6], among others.

For the linear elastodynamic wave equation in elastic anisotropic solids, three different waves, i.e., quasilongitudinal (*quasi-P*) and two transverse waves, are coupled together [5,6]. Because the quasi-P wave is of practical importance in obtaining maximal imaging resolution in seismic exploration [4], we develop theories for constructing the geometrical optics term (the leading

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term) in the asymptotic expansions for quasi-P waves in anisotropic solids, where the phase function is the so-called *traveltime*.

The eikonal equation is a first-order nonlinear Hamilton-Jacobi equation, and thus, can be solved by the method of characteristics which constructs the characteristic curves called "rays". The traveltime function and the amplitude function satisfy linear ordinary differential equations along the rays. The methods based on the characteristic equations are called "ray-tracing methods" [3,7,8], and they work for both *isotropic* and *anisotropic* solids. But ray-tracing methods have some drawbacks. The nonuniform distribution of traveltime data from ray-tracing methods gives rise to cumbersome and expensive interpolations for the application in seismic imaging.

The finite-difference eikonal solvers compute the viscosity solution corresponding to first-arrival times directly on a prespecified grid, involve rather simple data structures, and are easy to code efficiently [9–12], among others. However, extension of these methods to anisotropic wave propagations is not entirely straightforward. The finite-difference eikonal solvers mentioned above depend on the fact that for isotropic media the ray velocity vector, i.e., the *group velocity*, has the same direction as the traveltime gradient, i.e., the *phase velocity*, so that we can use the traveltime gradient as a reliable indicator of energy flow in extrapolating the traveltime field. However, this is not true for anisotropic media [13].

In [14], by making use of the convexity of quasi-P slowness surface, we established a reliable indicator of quasi-P ray velocity direction without computing the ray direction by formulating a relation between the group velocity direction and the phase velocity direction. The resultant eikonal equation is the so-called paraxial eikonal equation which has a built-in reliable indicator of the ray velocity direction and an $O(N)$ algorithm can be easily designed, where N is the number of spatial mesh points. Furthermore, complete algorithm details, second-order convergence, and illustrative applications are presented in [15] for solving the heterogeneous quasi-P paraxial eikonal equation by the Newton method and finite-difference methods. The above paraxial formulation provides us with a partial aperture of the traveltime field. However, it can be used to reformulate the anisotropic eikonal equation in any preferred direction so that we can obtain a full aperture eikonal solver based on the Fermat's least time principle; see [16] for details. In this paper, we present the theoretical justification for the paraxial formulation and prove the existence of the viscosity solution of the paraxial eikonal equation.

The paraxial formulation provides a way for deriving an evolution Hamilton-Jacobi equation from the stationary quasi-P eikonal equation so that high-order methods can be easily designed to obtain highly accurate solutions. There are other approaches such as a level-set formulation invented by Osher-Sethian [17], Osher [18]. However, the level-set formulation raises the Hamilton-Jacobi equation one dimension higher than the original equation; therefore, further work needs to be done to localize the level set formulation, especially for anisotropic eikonal equations. Recently, Sethian and Vladimirsky [19] proposed ordered upwind methods for static Hamilton-Jacobi equations. The methods also require that the Hamiltonian be convex with respect to the gradient of the solution and are of first-order accuracy.

In Section 2, we derive the quasi-P eikonal equation, including a simple geometrical argument of the convexity of the quasi-P slowness surface. In Section 3, we comment on the viscosity solution of the stationary quasi-P eikonal equations. In Section 4, we present and prove the paraxial formulation for the quasi-P eikonal equation. In Section 5, we comment on the viscosity solution of the Cauchy problem of paraxial quasi-P eikonal equations.

2. THE QUASI-P EIKONAL EQUATION

High frequency approximation to the elastic equation of motion leads to the Christoffel equation [6, p. 84],

$$\sum_k \left(\sum_{i,l} a_{ijkl} p_i p_l - \delta_{jk} \right) U_k = 0, \tag{1}$$

in which a_{ijkl} are the components of the elastic (Hooke) tensor divided by density, U_k is the displacement vector for a particular asymptotic phase, $\mathbf{p} = \nabla\tau$ is the slowness vector, τ is the traveltime or phase of the mode, and δ_{jk} is the Kronecker delta. Note that all of these quantities depend on the spatial coordinate vector $\mathbf{x} = (x_1, x_2, x_3)$, though in this and some of the following displays this dependence has been suppressed for the sake of clarity. This equation has nontrivial solutions U_k only when

$$\det\left(\sum_{i,l} a_{ijkl} p_i p_l - \delta_{jk}\right) = 0. \tag{2}$$

The Christoffel matrix $\sum_{i,l} a_{ijkl} p_i p_l$ is positive definite and scales as $p^2 = \mathbf{p}\cdot\mathbf{p}$. Therefore, each eigenvalue takes the form $v^2(\mathbf{x},\mathbf{p})p^2$, where v is a homogeneous function of degree zero in $\mathbf{p}$. The largest eigenvalue, denoted $v_{qP}^2(\mathbf{x},\mathbf{p})p^2$, is simple [5, p. 95], hence depends smoothly on the components of the Christoffel matrix, i.e., on $\mathbf{p}$, and the Hooke tensor. The slowness vector $\mathbf{p}$ for which

$$S(\mathbf{x},\mathbf{p}) \equiv |\mathbf{p}| v_{qP}(\mathbf{x},\mathbf{p}) = 1, \tag{3}$$

forms the qP *slowness surface*, and v_{qP} is the qP phase velocity. Note that these slowness vectors solve the Christoffel equation (2).

The qP wave eikonal equation results from combining the slowness surface condition (3) with the slowness identity $\mathbf{p} = \nabla\tau$

$$S(\mathbf{x}, \nabla\tau) = 1, \tag{4}$$

where S is continuous in $\mathbf{x}$ and $\mathbf{p}$.

The method of characteristics relates its solution τ of slowness surface equation (4) to the rays of geometrical optics, which are the solutions of the ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{p}} S(\mathbf{x},\mathbf{p}), \tag{5}$$

$$\frac{d\mathbf{p}}{dt} = -\nabla_{\mathbf{x}} S(\mathbf{x},\mathbf{p}), \tag{6}$$

where we have used the homogeneity of the eigenvalue v_{qP} in $\mathbf{p}$, so that $\tau = t$ has the dimension of time.

For every $\mathbf{x}$ the slowness surface $S = \{\mathbf{p} : S(\mathbf{x},\mathbf{p}) = 1\}$ is strictly *convex* in $\mathbf{p}$ by the following simple argument [6, p. 92]: the slowness surface defined by equation (2) is sextic and consists of three sheets corresponding to three different waves; if the inner detached slowness sheet related to quasi-P waves is not wholly strictly convex, a straight line could intersect the inner sheet at four or more points and yet make at least four further intersections with the remaining sheets; but any straight line must intersect the slowness surface at only six points, real or imaginary because the slowness surface is sextic.

A radiation problem is the problem where the solution is generated by a point source [2]. We will consider the radiation problem for the eikonal equation (4) in R^3 with the radiation source situated at a point $\mathbf{y}$. Thus, the radiation condition is imposed as

$$\tau|_{\mathbf{x}=\mathbf{y}} = 0. \tag{7}$$

Theoretically, the radiation solution plays the role as a fundamental solution for the eikonal equation. Such boundary condition suppresses the boundary data because the modulus of the gradient of the solution can be estimated directly from the eikonal equation in a neighborhood of the radiation point by a constant.

3. VISCOSITY SOLUTIONS OF STATIONARY EIKONAL EQUATIONS

We do not recall here the definition and basic properties of viscosity solutions and rather refer readers to [20].

For problem (4),(7), we introduce the following semidistance function between two arbitrary points $\mathbf{x}$ and $\mathbf{y}$,

$$L(\mathbf{x},\mathbf{y}) = \inf_{\boldsymbol{\xi}\in X_{\mathrm{ad}}} \left\{ \int_0^1 \max_{S(\boldsymbol{\xi}(t),\mathbf{p})=1} \left\{ -\left(\frac{d\boldsymbol{\xi}}{dt}, \mathbf{p} \right) \right\} dt \right\}, \tag{8}$$

where $\boldsymbol{\xi}$ is a path connecting $\mathbf{x}$ and $\mathbf{y}$ satisfying

$$X_{\mathrm{ad}} = \left\{ \boldsymbol{\xi}(t) \in R^3 : \boldsymbol{\xi}(0) = \mathbf{x},\ \boldsymbol{\xi}(1) = \mathbf{y},\ 0 \le t \le 1,\ \frac{d\boldsymbol{\xi}}{dt} \in L^\infty(0,1) \right\}. \tag{9}$$

Lions [21, p. 132, Theorem 5.3] proved that $L(\cdot,\mathbf{y})$ is the unique viscosity solution for equation (4) in $R^3 \setminus \{\mathbf{y}\}$ and satisfying (7). Since for each admissible path $\boldsymbol{\xi}$ connecting $\mathbf{x}$ and $\mathbf{y}$,

$$L(\boldsymbol{\xi}) = \int_0^1 \max_{S(\boldsymbol{\xi}(t),\mathbf{p})=1} \left\{ -\left(\frac{d\boldsymbol{\xi}}{dt}, \mathbf{p} \right) \right\} dt \tag{10}$$

is the optical length [21], the above result says that the viscosity solution gives the shortest path (or first-arrival times) among all the optical lengths, which is the Fermat principle. See [22] for a similar justification for the isotropic eikonal equation.

Even though the above recipe provides an explicit formula for computing the traveltime, it is not very convenient because we have to solve a minimax problem. To solve efficiently the above stationary eikonal equation with a Dirichlet boundary condition, a special case of Dirichlet problem for Hamilton-Jacobi (H-J) equations, using the convexity of the Hamiltonian we solve equation (4) for one of the normal derivatives in terms of the other two in such a way as to obtain a stable evolution equation in one direction.

4. THE QUASI-P PARAXIAL EIKONAL EQUATION

In velocity structures with mild lateral heterogeneity, most reflected wave energy propagates down to the target, then up to the surface. That is, the energy in such a wavefield propagates along downgoing rays: the x_3 ("z") component of the ray velocity vector remains positive from source to target. The traveltime along such downgoing rays increases with depth and should be the solution of an evolution system in depth. For isotropic wave propagation, Gray and May [23] suggested modifying the eikonal equation in such a way that

(i) the modified equation defines a depth evolution of traveltime, and
(ii) solutions of the modified and original eikonal equation are identical at every point connected to the source by a first-arriving ray making an angle with the vertical less than 90 degrees.

The principal purpose of this section is to justify such a modification, resulting in a *paraxial eikonal equation*, for anisotropic propagation.

In the following, we assume that the Hooke tensor is constant so that the slowness surface depends on $\mathbf{p}$ only and that the slowness surface S is C^3, bounded, closed and strictly convex.

LEMMA 4.1. *For any given $(p_1^*, p_2^*) \in \{(p_1,p_2) : \exists\, p_3 \Rightarrow S(p_1,p_2,p_3)=1\}$, draw the straight line passing through (p_1^*, p_2^*) and parallel to p_3 axis in the (p_1,p_2,p_3) space. Then there exist at most two intersection points (p_1^*, p_2^*, p_3^1) and (p_1^*, p_2^*, p_3^2). The outward normals at the two intersection points have p_3 components of opposite signs.*

PROOF. Consider 2-D case only; 3-D similar. By the strict convexity of the slowness surface, for p_1^* given, there are at most two intersection points. Without loss of generality, for p_1^* given, we can assume that there exist p_3^1 and p_3^2 such that $S(p_1^*, p_3^1) = S(p_1^*, p_3^2) = 0$; see Figure 1.

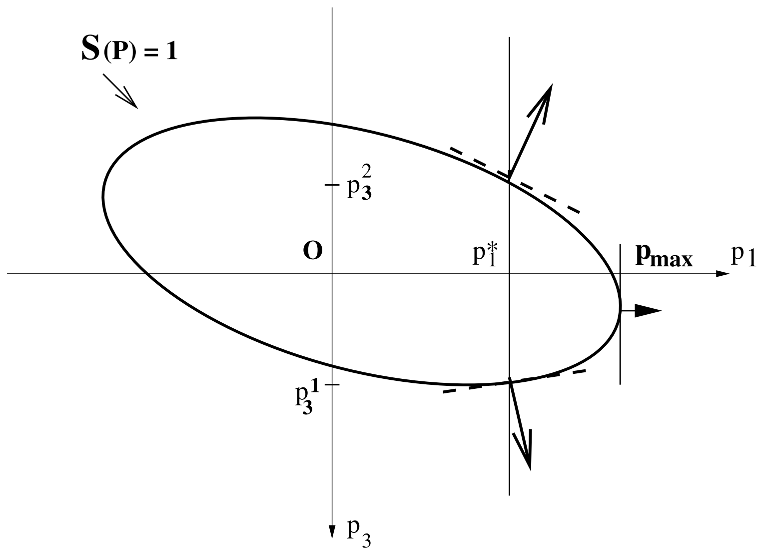

Figure 1. The p_3 components of outward normals at the two intersections on the convex slowness surface have opposite signs.

Let $p_3^1 > p_3^2$. For $p_3 \in [p_3^2, p_3^1]$, define two functions

$$f(p_3) = \sup\{\alpha : S(\alpha, p_3) = 1,\ \alpha \geq p_1^*\},$$
$$g(p_3) = \inf\{\alpha : S(\alpha, p_3) = 1,\ \alpha \leq p_1^*\}.$$

Since both f and g are well defined and convex (or concave) by convexity of S, one of them must satisfy that

$$f\left(p_3^1\right) = p_1^* = f\left(p_3^2\right)$$

or

$$g\left(p_3^1\right) = p_1^* = g\left(p_3^2\right).$$

Without loss of generality, supposing that f satisfies the above constraint, its graph corresponds to a section of slowness surface S.

By Rolle's mean theorem, there exists a $\xi \in (p_3^2, p_3^1)$ such that

$$f'(\xi) = 0.$$

However, f' is strictly monotonic by strict convexity of f, so

$$f'\left(p_3^1\right) f'\left(p_3^2\right) < 0.$$

Assuming that $f'(p_3^1) < 0$ and $f'(p_3^2) > 0$, they define the tangents at those two points. It follows that the outward normal at (p_1^*, p_3^1) has acute angle with positive p_3 direction, so it has positive p_3 component. Similarly, the outward normal at (p_1^*, p_3^2) has negative p_3 component.

By Lemma 4.1, the following function is well defined.

Definition 4.1. *For* $(p_1, p_2) \in \{(p_1, p_2) : \exists p_3 \Rightarrow S(p_1, p_2, p_3) = 1\}$, *define* $p_3 = H(p_1, p_2)$ *satisfying that*

$$S(p_1, p_2, H(p_1, p_2)) = 1, \qquad \frac{\partial S}{\partial p_3}(p_1, p_2, H(p_1, p_2)) \geq 0.$$

Downgoing rays correspond to the part of the slowness surface on which $\frac{dx_3}{dt} = S_{p_3} > 0$, thus Definition 4.1 defines p_3 as a function of $\mathbf{x}, p_1, p_2$

$$p_3 = H(\mathbf{x}, p_1, p_2), \tag{11}$$

which is also a partial differential equation for τ. The characteristics (rays) of eikonal equation (11) are downgoing, so they can be parameterized by $x_3 = z$ and satisfy

$$\frac{dx_i}{dx_3} = -\frac{\partial H}{\partial p_i}, \qquad i = 1, 2, \tag{12}$$

$$\frac{d\tau}{dx_3} = H - p_1 \frac{\partial H}{\partial p_1} - p_2 \frac{\partial H}{\partial p_2}. \tag{13}$$

However, neither the partial differential equation (11) nor its rays are well defined for all p_1, p_2. To remedy this defect, we introduce another family of Hamiltonian functions H_Δ, each of which is identical to H along "safely downgoing" rays, and is defined everywhere in the slowness space. Next, we first prove some results for 2-D cases, then we extend those results to 3-D cases by introducing horizontal coordinates.

LEMMA 4.2. 2-D CASE. *Suppose that $\partial\Omega = \{(p_1, p_3) : S(p_1, p_3) = 1\}$ is a bounded closed curve. Under the assumption of Lemma 4.1, the following two sets are both nonempty:*

$$F = \left\{p_1^* : \exists\, p_3^1,\ p_3^2 \Rightarrow f\left(p_3^1\right) = p_1^* = f\left(p_3^2\right)\right\},$$
$$G = \left\{p_1^* : \exists\, p_3^1,\ p_3^2 \Rightarrow g\left(p_3^1\right) = p_1^* = g\left(p_3^2\right)\right\},$$

where p_1^* satisfies that there exist p_3^1 and p_3^2 such that $S(p_1^*, p_3^1) = S(p_1^*, p_3^2) = 1$, and f and g are defined in Lemma 4.1. Moreover, $\inf F \le \sup G$.

PROOF. Obviously, at least one of them is nonempty. Without loss of generality, let F be nonempty. Supposing that G is empty, it means that all the $p_1 \in \{p_1 : \exists\, p_3 \Rightarrow S(p_1, p_3) = 1\}$ is in F. Because F is a closed bounded set, $\beta = \inf F > -\infty$. So for $p_1^* = \beta$, there exist p_3^1 and p_3^2 such that f' is strictly monotonic in $[p_3^2, p_3^1]$, hence $\partial\Omega$ cannot be closed, which is a contradiction; it follows that G is nonempty. Moreover, if $\inf F > \sup G$, $\partial\Omega$ consists of two branches which are not connected; therefore, it cannot be a closed convex curve. It follows that $\inf F \le \sup G$.

THEOREM 4.1. 2-D CASE. *Under the assumptions of Lemmas 4.1 and 4.2, among all (p_1, p_3) such that*

$$S(p_1, p_3) = 1, \qquad \frac{\partial S}{\partial p_3}(p_1, p_3) \ge 0,$$

there are $p_1^{\min}$, $p_1^{\max}$ such that

$$S\left(p_1^{\min}, H\left(p_1^{\min}\right)\right) = 1, \qquad \frac{1}{|\nabla S|}\frac{\partial S}{\partial p_3}\left(p_1^{\min}, H\left(p_1^{\min}\right)\right) = 0, \tag{14}$$

$$S\left(p_1^{\max}, H\left(p_1^{\max}\right)\right) = 1, \qquad \frac{1}{|\nabla S|}\frac{\partial S}{\partial p_3}\left(p_1^{\max}, H\left(p_1^{\max}\right)\right) = 0. \tag{15}$$

For $0 < \Delta < 1$ and $p_1 \in [(1-\Delta)p_1^{\min}, (1-\Delta)p_1^{\max}]$,

$$\frac{1}{|\nabla S|}\frac{\partial S}{\partial p_3}(p_1, H(p_1)) \ge O(\Delta) > 0. \tag{16}$$

PROOF. By Lemma 4.1 and $F \ne \emptyset$, for $\forall\, p_1^* \in F$, $\exists\, f$ such that $f' \le 0$ and f' is strictly decreasing in $[\xi, p_3^1]$, where $f'(\xi) = 0$. Define $p_1^{\max} = f(\xi)$, then $\xi = H(p_1^{\max})$, so (15) holds. By strict monotonicity of function f in $[\xi, p_3^1]$, for $p_1 = (1-\Delta)p_1^{\max}$, there exists $\eta \in (\xi, p_3^1)$ such that

$$f(\eta) = (1-\Delta)p_1^{\max}, \qquad \sigma = f'(\eta) < 0.$$

(Here, we assume $\xi \ne p_3^1$. The case $\xi = p_3^1$ can be treated similarly.) Then the outward normal at $(f(\eta), \eta)$ is $(1/\sqrt{1+\sigma^2}), (|\sigma|/\sqrt{1+\sigma^2}))$. Because f' is strictly decreasing, $f'(p_3) \le f'(\eta) < 0$ if $p_3 \ge \eta$. Furthermore, for $p_1 \in F$ and $p_1 \le (1-\Delta)p_1^{\max}$,

$$\frac{1}{|\nabla S(p_1, H(p_1))|}\frac{\partial S}{\partial p_3}(p_1, H(p_1)) \ge O(\Delta) > 0.$$

Similarly, by $G \neq \emptyset$, there exists $p_1^{\min}$ such that (14) holds. Also, for $p_1 \in G,\ p_1 \geq (1-\Delta)p_1^{\min}$,

$$\frac{1}{|\nabla S(p_1, H(p_1))|} \frac{\partial S}{\partial p_3}(p_1, H(p_1)) \geq O(\Delta) > 0.$$

Finally, by $\inf F \leq \sup G$ (Lemma 4.2), for $p_1 \in [(1-\Delta)p_1^{\min}, (1-\Delta)p_1^{\max}]$ we have

$$\frac{1}{|\nabla S(p_1, H(p_1))|} \frac{\partial S}{\partial p_3}(p_1, H(p_1)) \geq O(\Delta) > 0.$$

Definition 4.2. 1-D Hamiltonian H_Δ. *Given $\Delta > 0$, $p_1^{\min}$, $p_1^{\max}$ defined as in Theorem 4.1, for $\forall\, p_1 \in R$, define function H_Δ*

$$H_\Delta(p_1) = \begin{cases} H(p_1), & \text{if } p_1 \in [a, b], \\ H(a) + H_{p_1}(a)(p_1 - a), & \text{else if } p_1 < a, \\ H(b) + H_{p_1}(b)(p_1 - b), & \text{else if } p_1 > b, \end{cases} \tag{17}$$

where $a = (1-\Delta)p_1^{\min}$ and $b = (1-\Delta)p_1^{\max}$.

Theorem 4.2. *The Hamiltonian H_Δ is concave and C^1 in R.*

Proof. Only need to consider H_Δ in the interval $((1-\Delta)p_1^{\min}, (1-\Delta)p_1^{\max})$. By

$$S(p_1, p_3) = S(p_1, H(p_1)) = 1,$$

taking the derivative twice with respect to p_1 in the above, we have

$$\begin{pmatrix} 1 \\ H_{p_1} \end{pmatrix}^\top \begin{pmatrix} S_{p_1p_1} & S_{p_1p_3} \\ S_{p_1p_3} & S_{p_3p_3} \end{pmatrix} \begin{pmatrix} 1 \\ H_{p_1} \end{pmatrix} + S_{p_3} H_{p_1p_1} = 0.$$

Because the Hessian matrix in the above is positive definite by S strictly convex and $S_{p_3} > 0$, we have $H_{p_1p_1} < 0$. It follows that H is concave in $((1-\Delta)p_1^{\min}, (1-\Delta)p_1^{\max})$. Since outside $((1-\Delta)p_1^{\min}, (1-\Delta)p_1^{\max})$ H_Δ is a linear extension, it is concave.

Theorem 4.3. 3-D Case. *Given $\Delta > 0$, for any radial plane $p_2 = kp_1$ cutting through slowness surface S, there exist two intersection points*

$$\left(p_1^{\min}, kp_1^{\min}\right) = \left(p_1^{\min}(k), kp_1^{\min}(k)\right), \qquad \left(p_1^{\max}, kp_1^{\max}\right) = \left(p_1^{\max}(k), kp_1^{\max}(k)\right) \tag{18}$$

such that

$$\begin{aligned} S\left(p_1^{\min}, kp_1^{\min}, H\left(p_1^{\min}, kp_1^{\min}\right)\right) &= 1, \\ \frac{1}{|\nabla S|} \frac{\partial S}{\partial p_3}\left(p_1^{\min}, kp_1^{\min}, H\left(p_1^{\min}, kp_1^{\min}\right)\right) &= 0, \\ S\left(p_1^{\max}, kp_1^{\max}, H\left(p_1^{\max}, kp_1^{\max}\right)\right) &= 1, \\ \frac{1}{|\nabla S|} \frac{\partial S}{\partial p_3}\left(p_1^{\max}, kp_1^{\max}, H\left(p_1^{\max}, kp_1^{\max}\right)\right) &= 0. \end{aligned}$$

For $p_1 \in [(1-\Delta)p_1^{\min}(k), (1-\Delta)p_1^{\max}(k)]$ and $p_2 = kp_1$,

$$\frac{1}{|\nabla S|} \frac{\partial S}{\partial p_3}(p_1, p_2, H(p_1, p_2)) \geq O(\Delta) > 0. \tag{19}$$

Proof. When $k = 0$ or $k = \infty$, the theorem reduces to the 2-D case, which is proved in Theorem 4.1. Otherwise, we can rotate the axes of the coordinate by orthogonal transformation

$$M = \begin{pmatrix} \frac{1}{\sqrt{1+k^2}} & \frac{k}{\sqrt{1+k^2}} & 0 \\ -\frac{k}{\sqrt{1+k^2}} & \frac{1}{\sqrt{1+k^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that the new op'_1 lies in the plane $p_2 = kp_1$ and op'_2 is perpendicular to it. Thus, we have reduced the 3-D problem to a 2-D problem in $op'_1 p_3$ coordinate plane. By Theorem 3, there exists ${p'}_1^{\min}$ and ${p'}_1^{\max}$ such that

$$S'\left({p'}_1^{\min}, H'\left({p'}_1^{\min}\right)\right) = 1, \qquad \frac{1}{|\nabla S'|}\frac{\partial S'}{\partial p_3}\left({p'}_1^{\min}, H'\left({p'}_1^{\min}\right)\right) = 0, \tag{20}$$

$$S'\left({p'}_1^{\max}, H'\left({p'}_1^{\max}\right)\right) = 1, \qquad \frac{1}{|\nabla S'|}\frac{\partial S'}{\partial p_3}\left({p'}_1^{\max}, H'\left({p'}_1^{\max}\right)\right) = 0, \tag{21}$$

where S' and H' are intersections of the slowness surface S and Hamiltonian H with $op'_1 p_3$ plane (or radial plane $p_2 = kp_1$), respectively.

Performing the inverse transform to pull the above result back to original coordinates, we have an interval $[p_1^{\min}, p_1^{\max}]$, where $p_1^{\min} = (1/\sqrt{1+k^2})\, {p'}_1^{\min}$ and $p_1^{\max} = (1/\sqrt{1+k^2})\, {p'}_1^{\max}$, such that for $p_2 = kp_1$ the theorem holds.

It is convenient to parameterize the horizontal plane (p_1, p_2) by polar coordinates $(p_1, p_2) = (p\cos\phi, p\sin\phi)$ with $p = \sqrt{p_1^2 + p_2^2}$, and we can summarize Theorem 4.3 into a corollary.

COROLLARY 4.1. *For each planar angle ϕ, the family of planes with outward normal $(\cos\phi, \sin\phi, 0)$ is tangent to the slowness surface at exactly one point*

$$(p_{\max}(\phi)\cos\phi, p_{\max}(\phi)\sin\phi, p_3(\phi))$$

and satisfies that

$$S(p_{\max}(\phi)\cos\phi, p_{\max}(\phi)\sin\phi, p_3(\phi)) = 1,$$
$$\frac{1}{|\nabla S|}\frac{\partial S}{\partial p_3}(p_{\max}(\phi)\cos\phi, p_{\max}(\phi)\sin\phi, p_3(\phi)) = 0.$$

For $0 < \Delta < 1$, $p \le (1-\Delta)p_{\max}(\phi)$, and $(p_1, p_2) = (p\cos\phi, p\sin\phi)$, we have

$$\frac{1}{|\nabla S|}\frac{\partial S}{\partial p_3}(p_1, p_2, H(p_1, p_2)) \ge O(\Delta) > 0. \tag{22}$$

REMARK. By method of characteristics, we can see that Theorem 4.3 just puts a positive lower bound for the third component of a normalized ray direction; namely, the group angle of the ray with the vertical axis never goes near 90 degrees.

Next, we introduce the paraxial Hamiltonian which is defined in the whole horizontal slowness space.

DEFINITION 4.3. 2-D H_Δ. *Given $\Delta > 0$, for any $(p_1, p_2) = (p\cos\phi, p\sin\phi)$, there exists $p_{\max}(\phi)$ as in Corollary 4.1, therefore we define function H_Δ*

$$H_\Delta(p_1, p_2) = \begin{cases} H(p_1, p_2), & \text{if } p \le (1-\Delta)p_{\max}(\phi), \\ H(a_1, a_2) + \nabla H^\top(a_1, a_2)v, & \text{else,} \end{cases} \tag{23}$$

where $a_1 = (1-\Delta)p_{\max}(\phi)\cos\phi$, $a_2 = (1-\Delta)p_{\max}(\phi)\sin\phi$, $v^\top = (p_1 - a_1, p_2 - a_2)$, $\bar{\mathbf{p}}^\top = (p_1, p_2)$.

THEOREM 4.4. *The 2-D Hamiltonian H_Δ is C^1 and concave in R^2.*

PROOF. By definition, the H_Δ is C^1 in R^2. Taking second-order derivatives of S with respect to p_1 and p_2, we have that

$$S_{p_1p_1} + S_{p_1p_3}H_{p_1} + S_{p_1p_3}H_{p_1} + S_{p_3p_3}H_{p_1}^2 + S_{p_3}H_{p_1p_1} = 0, \tag{24}$$

$$S_{p_1p_2} + S_{p_1p_3}H_{p_2} + S_{p_2p_3}H_{p_1} + S_{p_3p_3}H_{p_1}H_{p_2} + S_{p_3}H_{p_1p_2} = 0, \tag{25}$$

$$S_{p_2p_2} + S_{p_2p_3}H_{p_1} + S_{p_2p_3}H_{p_2} + S_{p_3p_3}H_{p_2}^2 + S_{p_3}H_{p_2p_2} = 0. \tag{26}$$

First of all, by Theorem 4.2, we have $H_{p_1p_1} \le 0$, $H_{p_2p_2} \le 0$. Next, we prove that $H^2_{p_1p_2} \le H_{p_1p_1}H_{p_2p_2}$ so that the matrix $-(H_{p_ip_j})$ $(1 \le i, j \le 2)$ is positive definite. For arbitrary real y, by (24) times y^2 plus (25) times $2y$ plus (26), we have

$$w^\top A w + S_{p_3}\left(y^2 H_{p_1p_1} + 2yH_{p_1p_2} + H_{p_2p_2}\right) = 0, \tag{27}$$

where $w^\top = (y, 1)$ and

$$A = \begin{pmatrix} S_{p_1p_1} + 2S_{p_1p_3}H_{p_1} + S_{p_3p_3}H^2_{p_1} & S_{p_1p_2} + 2S_{p_1p_3}H_{p_2} + S_{p_3p_3}H_{p_1}H_{p_2} \\ S_{p_1p_2} + 2S_{p_2p_3}H_{p_1} + S_{p_3p_3}H_{p_1}H_{p_2} & S_{p_2p_2} + 2S_{p_2p_3}H_{p_2} + S_{p_3p_3}H^2_{p_2} \end{pmatrix}.$$

Taking transpose of expression (27) plus (27) itself,

$$\frac{1}{2} w^\top \left(A + A^\top\right) w + S_{p_3}\left(y^2 H_{p_1p_1} + 2yH_{p_1p_2} + H_{p_2p_2}\right) = 0. \tag{28}$$

Furthermore, we have that

$$\begin{aligned} w^\top B^\top S_{\mathbf{pp}} B w + S_{p_3}\left(y^2 H_{p_1p_1} + 2yH_{p_1p_2} + H_{p_2p_2}\right) = 0, \\ v^\top S_{\mathbf{pp}} v + S_{p_3}\left(y^2 H_{p_1p_1} + 2yH_{p_1p_2} + H_{p_2p_2}\right) = 0, \end{aligned}$$

where $S_{\mathbf{pp}}$ is the Hessian matrix, $v^\top = (y, 1, yH_{p_1} + H_{p_2})$ and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ H_{p_1} & H_{p_2} \end{pmatrix}.$$

Since $S_{\mathbf{pp}}$ is positive definite and $S_{p_3} > 0$, we conclude that

$$y^2 H_{p_1p_1} + 2yH_{p_1p_2} + H_{p_2p_2} < 0.$$

For the above inequality to hold, the discriminant $(H_{p_1p_2})^2 - H_{p_1p_1}H_{p_2p_2}$ must be negative, which is desired.

Corollary 4.2. *Under the assumption of Corollary 4.1, the inequality*

$$\sqrt{\left(\frac{\partial H_\Delta}{\partial p_1}\right)^2 + \left(\frac{\partial H_\Delta}{\partial p_2}\right)^2} \le O\left(\frac{1}{\Delta}\right)$$

holds.

Proof. By using ray equations (5), (12), and (13), we have

$$\frac{\partial S}{\partial p_i} = -\frac{\frac{\partial H}{\partial p_i}}{H - p_1\left(\frac{\partial H}{\partial p_1}\right) - p_2\left(\frac{\partial H}{\partial p_2}\right)}$$

for $i = 1, 2$ and

$$\frac{\partial S}{\partial p_3} = \frac{1}{H - p_1\left(\frac{\partial H}{\partial p_1}\right) - p_2\left(\frac{\partial H}{\partial p_2}\right)}.$$

The inequality follows by using Corollary 4.1.

Remark. Corollary 2 says that the ray angle ψ stays safely away from 90 degrees

$$\begin{aligned} |\tan\psi| &= \frac{\sqrt{\left(\frac{\partial x_1}{\partial \tau}\right)^2 + \left(\frac{\partial x_2}{\partial \tau}\right)^2}}{\frac{\partial x_3}{\partial \tau}} \\ &= \sqrt{\left(\frac{\partial H_\Delta}{\partial p_1}\right)^2 + \left(\frac{\partial H_\Delta}{\partial p_2}\right)^2} \le O\left(\frac{1}{\Delta}\right). \end{aligned}$$

Finally, we have a *paraxial eikonal equation*

$$\frac{\partial \tau}{\partial x_3} = H_\Delta\left(\frac{\partial \tau}{\partial x_1}, \frac{\partial \tau}{\partial x_2}\right), \tag{29}$$

where the H_Δ is concave in horizontal slowness space.

5. VISCOSITY SOLUTIONS OF PARAXIAL EIKONAL EQUATIONS

Since we are given a point source radiation condition (7), we have to generate an initial condition for the paraxial eikonal equation (29). Because the solution for the eikonal equation is usually unbounded, for example, the distance function unbounded in an unbounded domain, it is impractical to assume that the initial function is bounded and uniformly continuous as usually done [20].

Suppose we are set to solve the paraxial eikonal equation (29) with the radiation condition at the origin

$$\tau|_{\mathbf{x}=0} = 0. \tag{30}$$

Classical theory of the method of characteristics guarantees that a unique smooth solution exists locally at the origin for the equation (29). To compute the viscosity solution numerically by an explicit numerical method, for example, [24,25], we have to generate an initial condition on a manifold of dimension 2 from the radiation condition, which is a condition on a manifold of dimension 0 only. In other words, we have to generate a solution $\tau(\cdot,\cdot,x_3)$ at some depth $x_3 = x_3^0$ with $x_3^0 > 0$ small enough, which may not be bounded. By the definition of $L(\cdot, y)$ presented above, such solution always exists as long as the paraxial parameter $\Delta > 0$ is taken small enough. In the practical computation, of course, we have to work on a bounded domain; in order to generate the initial condition, we may use the classical ray tracing method [3] or a shooting method for two-point boundary value problems.

Thus, we assume that an initial condition is given as

$$\tau(x_1, x_2, 0) = \tau_0(x_1, x_2) \tag{31}$$

and consider the Cauchy problem (29),(31) in the strip $\pi_Z = \{0 < x_3 \le Z,\ -\infty < x_1,\ x_2 < \infty\}$.

The initial function $\tau_0(x_1, x_2)$ is only assumed to be continuous; however, our paraxial formulation (Corollary 4.2) guarantees that the characteristic speed is bounded, so that the domain of dependence on initial conditions is compact for any finite depth $x_3 = z$. By suitable modifications of Proposition 1 of [26], we can conclude that under the assumption of τ_0 continuous only, there exists a unique and stable viscosity solution $\tau(x_1, x_2, x_3)$ of the paraxial eikonal equation in the strip π_Z. In [20], similar conclusions are obtained under the assumption that the initial condition is uniformly bounded and continuous.

6. CONCLUSION

We have formulated the paraxial eikonal equation satisfied by the first-arrival traveltime associated with the qP wave propagation. This formulation provides a fast and efficient way for computing the viscosity solution of the resultant eikonal equation. Based on this formulation, high-order methods can be easily designed [14,15,25]. We proved the validity of this formulation and concluded that the (unbounded) viscosity solution of the paraxial eikonal equation exists and is unique.

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