Level set based Eulerian methods for multivalued traveltimes in both isotropic and anisotropic media

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Summary

We summarize level-set based Eulerian methods for multivalued traveltimes in both isotropic and anisotropic media. A self-intersecting wavefront in a two-dimensional (2-D) position space could be lifted up to be a smooth curve in a three-dimensional (3-D) space. The lifted curve in the 3-D space could be represented as the common zeros of two level set functions. The evolution of the 3-D curve could be simulated by evolving the two level set functions with appropriate velocity fields. The projection of the evolved 3-D curve to the original position space yields the (self-intersecting) wavefront at a desired moment. Numerical experiments including a sinusoidal waveguide model, the well-known Marmousi model and a transversely isotropic model, illustrate the effectiveness, accuracy and stability of the new level-set based Eulerian methods.

Introduction

Seismic inverse scattering theory is based upon geometric optics for modeling high frequency wave propagation (Symes, 1995). To characterize the leading singularity of the wave field, there need two functions, namely, traveltime and amplitude (Keller and Lewis, 1995). The traveltime function satisfies the so-called eikonal equation which is a first-order nonlinear partial differential equation. In general, the eikonal equation admits more than one Lipschitz continuous solutions (weak solutions); however, one of the physically relevant solutions, the so-called viscosity solution, is unique, which corresponds to the first-arrival traveltime for the eikonal equation (Crandall and Lions, 1983). Moreover, monotone finite difference methods applied to the eikonal equation are able to compute this viscosity-solution based first-arrival traveltime accurately and efficiently; see (Osher and Shu, 1991; van Trier and Symes, 1991; Qian and Symes, 2002a) for references therein.

Nevertheless, the first-arrival wavefront may not carry the most energetic part of the wave-field, and the later-arrival wavefronts may be more interesting and useful for modern high resolution seismic imaging (Liu and Bleistein, 1995). The classic ray tracing method yields both first arrivals and later arrivals, and works for both isotropic and anisotropic media; however, it suffers from some shortcomings; for example, the traveltime is not uniformly distributed to the computational domain and the related interpolation onto a uniform mesh is cumbersome and expensive, although there are some improvements in this regard at the cost of complicated data structure and bookkeeping (Vinje et al., 1993). On the other hand, finite-difference based first arrivals only eikonal solvers could be designed to work for both isotropic and anisotropic media with traveltimes distributed uniformly to the computational domain; see (Qian and Symes, 2002b) and references therein. Thus, the challenge is to design finite difference based Eulerian methods for multivalued traveltimes which are able to produce uniformly distributed all arrival traveltimes, including both first and later arrivals, and work for both isotropic and anisotropic media.

In principle, there are two kinds of multivaluedness for the traveltime field. One is due to the inhomogeneity of the material parameters, and the other is due to the structure of the material. In isotropic media, the multivaluedness of the traveltime field is only due to the former and the singularity of the wavefront needs some time to develop during the wave propagation; most of the current available multivalued traveltime solvers only deal with this kind of singularity (Engquist et al., 1995; Benamou, 1996; Symes, 1996; Engquist et al., 2001). In anisotropic media, the multivaluedness of the traveltime field may be caused by both the inhomogeneity and the structure of the material; for example, the instantaneous singularity associated with the quasi-transverse (qS) wave in a transversely isotropic (TI) medium is caused by the structure and exists even in a homogeneous TI medium. Recently, Qian et al (Qian et al., 2001) have extended the level set based, phase space ray tracing Eulerian approach for isotropic media, first proposed in Osher et al (Osher et al., 2001), to anisotropic media, and the approach is very successful in computing the multivalued traveltimes in both isotropic and anisotropic media and yields uniformly distributed traveltime fields. Fomel and Sethian (Fomel and Sethian, 2001) have also proposed a so-called fast phase space method for multivalued traveltimes, but so far their numerical examples are only for isotropic media.

The level set method was originally designed for problems dealing with codimension one object, where it has been extremely successful, especially when topological changes in the interface (Osher and Sethian, 1988), and it has found its ways into many different applications (Sethian, 1996). Motivated by its successfulness for codimension one objects, Cheng (Cheng, 2000) has implemented a vectorial level set approach for capturing codimension two objects in 3-D spaces; namely, a curve in a 3-D space. The apparent advantage of the vectorial level set formulation is that it provides us with an Eulerian PDE framework so that a 3-D curve is well represented and resolved in a 3-D space. A self-intersecting wavefront in a 2-D space is a codimension one object in the 2-D space, and as such a single level set function in the 2-D space is not appropriate for capturing those self-intersecting wavefronts. However, the self-intersecting wavefront may be lifted up to be a smooth curve in a 3-D space. This leads us to use the vectorial level set formulation to move the lifted wavefront in the 3-D space according to some velocity field until some desired time. Then we project the 3-D curve at that time onto the original 2-D space, and thus it gives us the desired self-intersecting wavefront.

First we present the vectorial level set methodology for moving curves in 3-D spaces. Then we outline how to obtain reduced phase space ray tracing systems for both isotropic and anisotropic media. Next we give some implementation details for evolving the two level set functions. Extensive numerical experiments illustrate the effectiveness, accuracy and stability of the new level-set based Eulerian methods.

The level set methodology

In the vectorial level set formulation for a curve in a 3-D space, the curve is represented by the intersection between the zero level sets of two level set functions, ϕ and ψ, i.e., where $\phi = \psi = 0$. Under this representation, moving a curve by a certain type of motion is accomplished by evolving the two functions ϕ and ψ in a 3-D space, and the intersection of their zero level sets gives the desired curve.

Let $\Gamma(t) = \{\gamma(s,t) = (x(s,t), y(s,t), z(s,t) : 0 \leq s \leq 1, t \geq 0\}$ be the curve at time t, which is defined by the intersection of two zero level sets

$$\Gamma(t) = \{\gamma(s,t) : \phi(t,\gamma(s,t)) = \psi(t,\gamma(s,t)) = 0, 0 \leq s \leq 1\}$$

Differentiating with respect to t in $\phi(t,\gamma(s,t)) = 0$ and $\psi(t,\gamma(s,t)) = 0$ yields

$$\nabla\phi(t,\gamma(s,t))\cdot\gamma_t(s,t) + \phi_t(t,\gamma(s,t)) = 0$$

$$\nabla\psi(t,\gamma(s,t))\cdot\gamma_t(s,t) + \psi_t(t,\gamma(s,t)) = 0$$

Given a velocity field $W = \gamma_t(s,t) \equiv (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$, by standard level set theory the PDE

$$\nabla\phi\cdot\gamma_t + \phi_t = 0$$

moves the zero level sets of ϕ according to $\gamma_t = W$. Therefore, the above PDE system moves the curve defined by intersections of the zero level sets of ϕ and ψ according to $\gamma_t = W$. We refer the reader to Cheng (2000) for more details.

Ray tracing system in reduce phase spaces

The 2-D eikonal equation in an isotropic medium states that

$$|\nabla\tau| = \frac{1}{v}$$

where $v = v(x,z)$ is the velocity. We define Hamiltonian $H(\mathbf{x},\mathbf{p}) = \frac{1}{2}(v^2|\nabla\tau|^2 - 1)$, where $\mathbf{x} = (x,z)$ and $\mathbf{p} = \nabla\tau$. Then by the method of characteristics, we have

$$\frac{d\mathbf{x}}{dt} = v^2\mathbf{p}, \frac{d\mathbf{p}}{dt} = -\frac{1}{v}\nabla_{\mathbf{x}}v$$

Introducing phase angle θ and parameterizing $\mathbf{p}$ by θ: $p_1 = \frac{\sin\theta}{v}$ and $p_2 = \frac{\cos\theta}{v}$, we have

$$\frac{dx}{dt} = v\sin\theta, \frac{dz}{dt} = v\cos\theta$$

$$\frac{d\theta}{dt} = v_z\sin\theta - v_x\cos\theta$$

Define $W = (\frac{dx}{dt}, \frac{dz}{dt}, \frac{d\theta}{dt})$, which is desired velocity field for the level set motion system.

Since W is defined by a reduced ray tracing system, the level set motion using the above velocity field as the passive velocity field could be understood as an Eulerian, reduced phase space ray tracing. However, different from the traditional ray tracing, the new approach yields wavefronts which are uniformly distributed to the computational domain. Another point worth mentioning is that the eikonal equation is nonlinear while the level set motion equations are linear. The stability condition for finite difference methods applied to the linear advection equation is easy to be satisfied and guarantee the convergence of the scheme by Lax Equivalence Theorem; however, the CFL stability condition for the eikonal equation is only a necessary condition for convergence.

Similarly, we can derive reduced phase space ray tracing system for anisotropic media (Qian et al., 2001). Consider a 2-D anisotropic eikonal equation defined by

$$F(x,z,\tau_x,\tau_z) = F(x,z,p_1,p_3) = 0$$

Parameterize the slowness vector by

$$p_1 = \frac{\cos\theta}{V(x,z,\theta)}, p_3 = \frac{\sin\theta}{V(x,z,\theta)}$$

where θ is the so-called phase angle varying from $-\pi$ to π and V the phase velocity depending on θ. Then the method of characteristics gives us

$$\frac{dx_1}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_2\frac{\partial F}{\partial p_2}\right)^{-1}\frac{\partial F}{\partial p_1}$$

$$\frac{dx_2}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_2\frac{\partial F}{\partial p_2}\right)^{-1}\frac{\partial F}{\partial p_2}$$

$$\frac{d\theta}{dt} = \left(p_1\frac{\partial F}{\partial p_1} + p_2\frac{\partial F}{\partial p_2}\right)^{-1}\left(V\frac{\partial F}{\partial x_1}\sin\theta - V\frac{\partial F}{\partial x_2}\cos\theta\right)$$

The above is the velocity field W needed for the level set motion equations. See Qian et al (2001) for more details.

Implementations

Since the two advection equations in the level set motion system are decoupled, we only need to consider the solution for one of them; the other follows the similar strategy. For example, to solve $\phi_t + W \cdot \nabla\phi = 0$ by finite difference methods, we may use high-order finite difference schemes for Hamilton-Jacobi equations, such as third-order Weighted Essentially Non-Oscillatory (WENO) Godunov scheme (Jiang and Peng, 2000) for spatial derivatives, and third order Total Variation Diminishing (TVD) Runge-Kutta scheme for time derivatives (Osher and Shu, 1991); see also (Qian and Symes, 2002a) for the application of this scheme to eikonal and advection equations.

For the explicit scheme applied to the linear advection equation, the stability condition states that the time step Δt must be

$$\Delta t \leq C \frac{\Delta x}{\max |W|}$$

where $\Delta x = \Delta z = \Delta\theta$ is the spatial step for a uniform mesh, and C is a Courant-Friedrichs-Levy constant which may be chosen to be less than 1.

In some situations, the appearance of kinks in the level set functions may affect the PDE solvers, therefore reinitialization steps are used to regularize the level set functions so that the zero level sets of ϕ and ψ are orthogonal to each other at their points of intersection. See (Osher et al., 2001) for more details.

One last important point is how to obtain the intersection of the zero level sets of the ϕ and ψ. Each small cube in the 3-D mesh is broken up into six tetrahedra, inside of which ϕ and ψ can be approximated by hyperplanes. The intersection of the zero level sets of the two hyperplanes can then be computed, giving a small segment of line inside each tetrahedron. The union of all these segments gives an approximation of the curve. See also (Cheng, 2000) for more.

Numerical experiments

To illustrate the effectiveness of the level set based Eulerian method, we apply the method to several models.

The first is an anisotropic model, the Greenriver Shale. Figure 1 is the computed qS wavefronts for the model which is a transversely isotropic model; the singularity for qS wavefronts is due to the non-convexity of the corresponding slowness surface. See (Qian et al., 2001) for more.

The next model is a sinusoidal waveguide model (Symes, 1996), and its velocity is given by

$$v(x, z) = 1 + 0.2 \sin 0.5\pi z \sin 3\pi x$$

The source point is located at $(0.55, 0.05)$ near the center of the top of the model. Slow regions form lenses and create crossing rays, imperfect foci, and caustics (Symes, 1996). Figure 2 is the traveltime wavefronts including all arrivals computed by the level-set approach.

The last example is the well-known Marmousi model. Figure 3 is the traveltime wavefronts including all arrivals by the level set approach, where the source is located at $(6, 2.808)$ km.

Conclusions

We have summarized the vectorial level set based Eulerian approach for performing phase space ray tracing, and we have verified that the approach is able to capture both developing singularity and instantaneous singularity appearing in the wave propagation; especially the approach is applicable to both isotropic and anisotropic eikonal equations.

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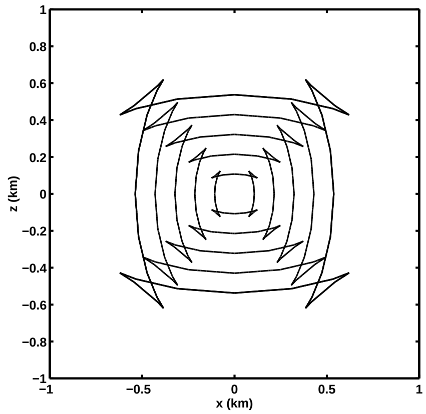

Fig. 1: The traveltime wavefronts for qS waves in Greenriver shale.

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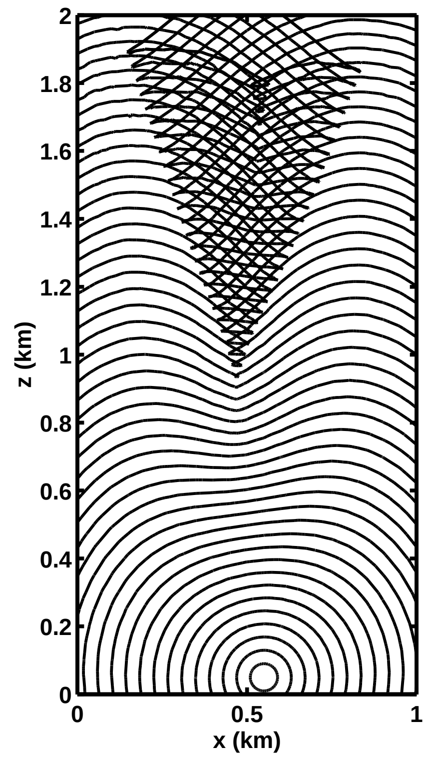

Fig. 2: The traveltime wavefronts for a sinusoidal model: all arrivals

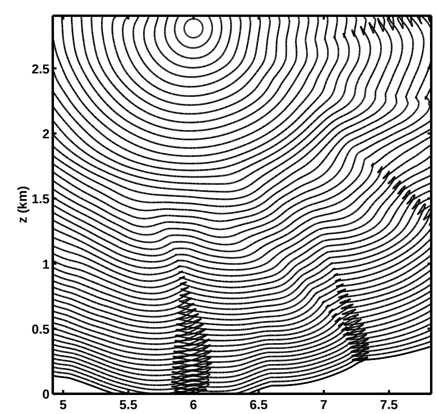

Fig. 3: The traveltime wavefronts for the Marmousi model: a windowed portion of the original model; the energy is carried upon later arrivals.