EXTENDING BABICH’S ANSatz FOR POINT-SOURCE MAXWELL’S EQUATIONS USING HADAMARD’S METHOD∗

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Abstract. Starting from Hadamard’s method, we extend Babich’s ansatz to the frequency-domain point-source (FDPS) Maxwell’s equations in an inhomogeneous medium in the high-frequency regime. First, we develop a novel asymptotic series, dubbed Hadamard’s ansatz, to form the fundamental solution of the Cauchy problem for the time-domain point-source (TDPS) Maxwell’s equations in the region close to the source. Governing equations for the unknowns in Hadamard’s ansatz are then derived. In order to derive the initial data for the unknowns in the ansatz, we further propose a condition for matching Hadamard’s ansatz with the homogeneous-medium fundamental solution at the source. Directly taking the Fourier transform of Hadamard’s ansatz in time, we obtain a new ansatz, dubbed the Hadamard–Babich ansatz, for the FDPS Maxwell’s equations. Next, we elucidate the relation between the Hadamard–Babich ansatz and a recently proposed Babich-like ansatz for solving the same FDPS Maxwell’s equations. Finally, incorporating the first two terms of the Hadamard–Babich ansatz into a planar-based Huygens sweeping algorithm, we solve the FDPS Maxwell’s equations at high frequencies in the region where caustics occur. Numerical experiments demonstrate the accuracy of our method.

Key words. Hadamard’s method, Babich’s ansatz, Maxwell’s equation, caustics, fast Huygens sweeping, high-frequency waves

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1. Introduction. We consider the following frequency-domain point-source (FDPS) Maxwell’s equations in \( \mathbb{R}^3 \):

\[
\nabla \times \nabla \times \mathbf{G}(\mathbf{r} ; \mathbf{r}_0) - k_0^2 n^2(\mathbf{r}) \mathbf{G}(\mathbf{r} ; \mathbf{r}_0) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}_0),
\]

where \( \mathbf{G}(\mathbf{r} ; \mathbf{r}_0) \) is the so-called dyadic Green’s function at the source \( \mathbf{r}_0 \), \( \mathbf{I} \) is the \( 3 \times 3 \) identity dyad, \( k_0 = \frac{\omega}{c} \) is the freespace wavenumber, \( \omega \) is the frequency, \( c \) is the speed of light in vacuum, \( \mathbf{r} = (x_1, x_2, x_3)^T \), and \( n = n(\mathbf{r}) \) is the variable refractive index of the medium. Throughout this paper, we assume that \( n \) is smooth over \( \mathbb{R}^3 \) and \( 1 - n \) has a compact support. Tacitly, the Silver–Muller radiation condition is assumed at infinity.

The dyadic Green’s function \( \mathbf{G} \) is fundamental in many applications, ranging from optics, microwaves, and antennas to radars. For a generic inhomogeneous medium, an analytic form of \( \mathbf{G} \) is in general not available. Numerical methods such as finite-difference time-domain (FDTD) methods or finite-element methods (FEMs) have thus...
been developed to numerically compute $G$. However, such direct approaches become very costly at high frequencies since they cannot bypass the self-inflicted pollution effect [3]: in order to attain the same accuracy level, the number of mesh points per unit length in each spatial direction has to be superlinearly dependent on frequency. Therefore, we seek alternative methods, such as asymptotic methods or methods of geometrical optics (GO), to carry out scale separation to solve (1.1) when $\omega$ is large.

An intuitive approach is to use the following Wentzel–Kramers–Brillouin (WKB) GO ansatz

\begin{equation}
G(r; r_0) = \sum_{l=0}^{\infty} \bar{A}_l(r; r_0) \frac{e^{i\omega \tau(r; r_0)}}{\omega^l},
\end{equation}

where the unknowns $\bar{A}_l$ and $\tau$ are independent of $\omega$. Using the leading term of (1.2) to approximate $G$, we have developed an Eulerian GO-based Huygens sweeping method to solve (1.1) in [22]. There we have shown that keeping only the leading term of (1.2) fails to capture correct singularities of $G$ at $r = r_0$.

Moreover, such drawbacks cannot be easily resolved by using two or more terms in (1.2) since as in the situation of solving the (scalar) FDPS Helmholtz equation, a critical challenge is how to initialize $\bar{A}_l$ at the source $r_0$. To resolve these issues, Babich in [2] proposed an asymptotic series based on Hankel functions, dubbed Babich’s ansatz, to expand the highly oscillatory wavefield. This new ansatz yields a uniform asymptotic solution as $\omega \to \infty$ in the region of space containing the point source but no other caustics. It is worth mentioning that his method of finding such an ansatz is closely bound up with Hadamard’s method of forming the fundamental solution of the Cauchy problem for the time-domain point-source (TDPS) Helmholtz wave equation; details were given in [9] and then were outlined by Courant and Hilbert [5]. Recently, Babich’s ansatz has been applied to numerically solve the FDPS Helmholtz equation in [16, 25]. Nevertheless, Babich’s ansatz cannot be trivially extended to the FDPS Maxwell’s equations (1.1).

To address the issues mentioned above, in [15] we proposed for (1.1) a novel ansatz, dubbed the Babich-like ansatz, based on the spherical Hankel functions, i.e.,

\begin{equation}
G(r; r_0) = \sum_{l=0}^{\infty} A_l(r; r_0) \frac{e^{i\omega \tau(r; r_0)}}{\omega^l},
\end{equation}

where the unknowns $A_l$ and $\tau$ are independent of $k_0$ (and $\omega$). We have demonstrated that the Babich-like ansatz (1.3) gives a uniform asymptotic expansion of the underlying solution in the region of space containing a point source but no other caustics.

The motivations of the current article are the following two questions. First, considering the close relation between the Helmholtz equation and Maxwell’s equations, can we extend Hadamard’s method to produce an asymptotic series similar to Babich’s ansatz for the FDPS Maxwell’s equations (1.1)? Second, if such a series exists, is it closely related to our Babich-like ansatz (1.3)? As we shall see, both questions are answered affirmatively in this paper.

The paper is organized as follows. First, we apply Hadamard’s method to develop an asymptotic series, dubbed Hadamard’s ansatz, to form the fundamental solution of the Cauchy problem for the TDPS Maxwell’s wave equations in a region close to the source. Governing equations for the unknowns in Hadamard’s ansatz are then derived. By comparing Hadamard’s ansatz with the homogeneous-medium fundamental solution, we propose a matching condition at the source which in turn gives
the initial data for the unknowns. By taking the Fourier transform of Hadamard’s ansatz in time, we immediately obtain the extension of Babich’s ansatz, which we dub the Hadamard–Babich ansatz, for the FDPS Maxwell’s equations (1.1). Next, we elucidate the relation between the new Hadamard–Babich ansatz and the previously proposed Babich-like ansatz (1.3), demonstrating their equivalence. Incorporating the first two terms of the Hadamard–Babich ansatz into a planar-based Huygens sweeping algorithm in [22], we solve (1.1) when \( \omega \) is high and when caustics occur. Numerical experiments demonstrate the accuracy of the new method.

2. Hadamard’s method. As indicated in [2], to apply Hadamard’s method, we need consider the following Cauchy problem of the TDPS Maxwell’s equations:

\[
\begin{align*}
\nabla \times \nabla \times G(r, t; r_0) + s^2(r)\ddot{G}(r, t; r_0) &= I\delta(r - r_0)\delta(t), \\
G(r, t; r_0)|_{t<0} &= 0,
\end{align*}
\]

where \( s(r) = n(r)/c \) is the slowness function, and \( \ddot{G} \) denotes the second-order time derivative of \( G \). We seek the solution to (2.1) and (2.2) in terms of the (generalized) functions \( f^{(\lambda)}_+ \) defined to be

\[
f^{(\lambda)}_+(x) = \frac{x_+^{\lambda}}{\Gamma(\lambda + 1)}, \quad \text{with } x_+ = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}
\]

for \( \lambda > -1 \); for other values of \( \lambda \), \( f^{(\lambda)}_+ \) is defined by analytic continuation. Also, since

\[
f^{(\lambda)}_+' = f^{(\lambda-1)}_+,
\]

\( f^{(\lambda)}_+ \) can be defined for negative integer values of \( \lambda \) by successive differentiation in the sense of distributions. Since \( f^{(0)}_+(x) = H(x) \), the Heaviside unit function, \( f^{(-l-1)}_+(x) = \delta^{(l)}(x) \), the \( l \)-th derivative of the \( \delta \)-function for \( l = 0, 1, 2, \ldots \). For further discussion of \( f^{(\lambda)}_+ \) and related functions, see Chapter I and sections 3.4 and 3.5 of [8].

Equation (2.1) indicates that the current density satisfies

\[
-\mu_0 \frac{\partial J}{\partial t} = I\delta(r - r_0)\delta(t),
\]

which yields

\[
J(r, t; r_0) = -\frac{1}{\mu_0}I\delta(r - r_0)H(t),
\]

where \( \mu_0 \) denotes the magnetic permeability, assumed constant. This indicates that the \( j \)-th column of the dyadic Green’s function \( G \) is the electric field due to a current density produced by a sudden localized current at \( r_0 \) of magnitude \(-\frac{1}{\mu_0}\) along the \( j \)-th coordinate direction starting at \( t = 0 \) for \( j = 1, 2, 3 \).

Using the relation

\[
\nabla \times \nabla \times G = \nabla(\nabla \cdot G) - \Delta G,
\]

(2.1) becomes

\[
\nabla(\nabla \cdot G) - \Delta G + s^2(r)\ddot{G}(r, t; r_0) = I\delta(r - r_0)\delta(t)
\]
so that we may rewrite it in the form

\[
G_{kj,ki} - G_{ij,kk} + s^2 \ddot{G}_{ij} = \delta_{ij}(r - r_0)\delta(t),
\]

(2.8)

\[
G_{ij} = 0 \quad \text{for} \quad t < 0.
\]

(2.9)

Here, the subscript and comma notations are employed. Specifically, \(G_{ij}\) denotes the \(ij\)th entry of \(G\), and \(G_{ij,k}\) denotes the \(x_k\)-derivative of \(G_{ij}\), so that \(G_{ij,kl}\) denotes \(\partial^2 x_k x_l G_{ij}\) for \(1 \leq i, j, k, l \leq 3\). Furthermore, the Einstein summation convention is assumed so that \(G_{ij,kk} = \Delta G_{ij}\), and so on.

Using Duhamel’s principle (p. 202 and pp. 729–730 in [5]), we may rewrite (2.8) for \(t > 0\) as

\[
G_{kj,ki} - G_{ij,kk} + s^2 \ddot{G}_{ij} = 0,
\]

(2.10)

with initial conditions

\[
G_{ij}(r, 0; r_0) = 0,
\]

(2.11)

\[
\dot{G}_{ij}(r, 0; r_0) = \frac{1}{s_0} \delta_{ij}\delta(r - r_0),
\]

(2.12)

where \(s_0 = s(r_0)\) is the slowness at the source point \(r_0\).

2.1. Homogeneous medium. In the case when \(s(r) \equiv s_0\), we give a closed form for \(G\). Following a suggestion of Friedlander [7], let us seek a solution to (2.10)–(2.12) in the form

\[
G_{ij} = \phi_{,ij} + \delta_{ij}\psi_{,kk} - \psi_{,ij} = \delta_{ij}\psi_{,kk} + (\phi - \psi),_{ij}\]

(2.13)

where \(\phi\) and \(\psi\) are to be determined. On substituting this into (2.10), we obtain

\[
\partial_i \partial_j \left[s_0^2 \ddot{\phi}\right] + \left[\delta_{ij} \Delta - \partial_i \partial_j\right] \left[s_0^2 \ddot{\psi} - \Delta \psi\right] = 0.
\]

(2.14)

Let us now decompose the initial condition (2.12) in a similar way, using the fact that

\[
\delta(r - r_0) = \Delta \left(\frac{-1}{4\pi r}\right),
\]

(2.15)

where \(r = |r - r_0|\). Thus, we may write

\[
\delta_{ij}(r - r_0) = \partial_i \partial_j \left(\frac{-1}{4\pi r}\right) + \left(\delta_{ij} \Delta - \partial_i \partial_j\right) \left(\frac{-1}{4\pi r}\right).
\]

(2.16)

Consequently, we see that \(G_{ij}\) given by (2.13) will satisfy (2.10)–(2.12) if

\[
s_0^2 \ddot{\phi} = 0, \quad t > 0,
\]

(2.17)

\[
\phi = 0, \quad \dot{\phi} = \frac{-1}{4\pi s_0^2 r}, \quad t = 0,
\]

(2.18)

and

\[
s_0^2 \ddot{\psi} - \Delta \psi = 0, \quad t > 0,
\]

(2.19)

\[
\psi = 0, \quad \dot{\psi} = \frac{-1}{4\pi s_0^2 r}, \quad t = 0.
\]

(2.20)
We easily verify that (2.17)–(2.20) are satisfied by
\begin{equation}
\phi = \frac{-t}{4 \pi s_0^2 r},
\end{equation}
\begin{equation}
\psi = \frac{R(t - s_0 r)}{4 \pi s_0^2 r} - \frac{t}{4 \pi s_0^2 r} = \begin{cases} \frac{-1}{4 \pi s_0}, & s_0 r \leq t, \\ \frac{-t}{4 \pi s_0^2 r}, & s_0 r > t, \end{cases}
\end{equation}
where
\begin{equation}
R(\xi) = \begin{cases} \xi, & \xi > 0, \\ 0, & \xi \leq 0. \end{cases}
\end{equation}

Carrying out the differentiations indicated in (2.13), we finally arrive at
\begin{equation}
G_{ij} = \frac{t \delta_{ij} (r - r_0)}{s_0^2} + \frac{\delta_{ij} - \gamma_i \gamma_j}{4 \pi r} t - \frac{3 \gamma_i \gamma_j}{4 \pi s_0^2} \left[ s_0 \frac{1}{r^2} H(t - s_0 r) + \frac{1}{r^3} R(t - s_0 r) \right]
\end{equation}
where \( \gamma_i = r_i/r \) and \( r - r_0 = (r_1, r_2, r_3)^T \).

Now, we need figure out the relation between \( G_{ij} \) and the generalized functions \( f^+(t) \) in (2.3). It turns out that the second-order time derivative of \( G_{ij} \) has a simple relation with \( f^+(t) \) and \( f^+(t) \).

**Proposition 2.1.**

\begin{equation}
\ddot{G}_{ij} = \frac{2 s_0}{\pi} \left[ (s_0 r)^2 (\delta_{ij} - \gamma_i \gamma_j) f^+(t^2 - (s_0 r)^2) - \delta_{ij} f^+(t^2 - (s_0 r)^2) \right].
\end{equation}

**Proof.** To prove (2.25), we require the following properties of the \( \delta \)-function: For \( t, (s_0 r) \geq 0 \),
\begin{equation}
\delta^{(1)}(t^2 - (s_0 r)^2) = \frac{1}{4(s_0 r)^3} (\delta(t - s_0 r) + (s_0 r) \delta^{(1)}(t - s_0 r)),
\end{equation}
\begin{equation}
\delta^{(2)}(t^2 - (s_0 r)^2) = \frac{1}{8(s_0 r)^3} \delta^{(2)}(t^2 - (s_0 r)^2) + \frac{3}{2(s_0 r)^2} \delta^{(1)}(t^2 - (s_0 r)^2).
\end{equation}
They indicate that
\begin{equation}
\begin{align*}
\partial_t^2 (t H(t - s_0 r) &= 4(s_0 r)^3 \delta^{(1)}(t^2 - (s_0 r)^2), \\
\partial_t^2 (\delta(t - s_0 r)) &= \delta^{(2)}(t - s_0 r) = 8(s_0 r)^3 \delta^{(2)}(t^2 - (s_0 r)^2) \\
&- 12 s_0 r \delta^{(1)}(t^2 - (s_0 r)^2). 
\end{align*}
\end{equation}
Consequently, one obtains
\begin{align*}
\ddot{G}_{ij} &= \frac{(\delta_{ij} - \gamma_i \gamma_j)}{4 \pi s_0^3} \left( \delta(t - s_0 r) + (s_0 r) \delta^{(1)}(t - s_0 r) \right) + \frac{\delta_{ij} - \gamma_i \gamma_j}{4 \pi r} \delta^{(2)}(t - s_0 r) \\
&= \frac{2 s_0}{\pi} \left[ (s_0 r)^2 (\delta_{ij} - \gamma_i \gamma_j) f^+(t^2 - (s_0 r)^2) - \delta_{ij} f^+(t^2 - (s_0 r)^2) \right].
\end{align*}
2.2. Inhomogeneous medium. When \( s(r) \) varies with \( r \), Proposition 2.1 motivates us to seek the solution of (2.10) in the following form of asymptotic series:

\[
\ddot{G}(r, t; r_0) = \sum_{l=0}^{\infty} B^l(r; r_0) f_+^{(-3+l)}(t^2 - \mathcal{T}(r; r_0))
\]

or, in subscript notation,

\[
\ddot{G}_{ij} = \sum_{l=0}^{\infty} B^l_{ij} f_+^{(-3+l)}(t^2 - \mathcal{T}),
\]

where we enforce all dyadic coefficients \( B^l = O(1) \) as \( r \to 0^+ \). In the following, we will refer to the above series as Hadamard’s ansatz. Notice that due to the initial conditions (2.11) and (2.12), Hadamard’s ansatz uniquely determines \( G \). In a homogeneous medium, Hadamard’s ansatz terminates at \( l = 1 \) with

\[
\mathcal{T} = (s_0 r)^2,
\]

\[
B^0 = \frac{2s_0^3 r^2}{\pi} (I - \tilde{r} \tilde{r}^T),
\]

\[
B^1 = -\frac{2s_0}{\pi} I,
\]

where \( I \) is the \( 3 \times 3 \) identity matrix and \( \tilde{r} = (r - r_0)/r \). To derive the governing equations for \( B^k \) and \( \mathcal{T} \), we need the following two simple properties of \( f_+^{(k)} \).

**Proposition 2.2.** We have for all \( k \in \mathbb{Z}, x \in \mathbb{R} \) the following:

(a)

\[
f_+^{(k)}(x) = f_+^{(k-1)}(x).
\]

(b)

\[
x f_+^{(k)}(x) = (k + 1) f_+^{(k+1)}(x).
\]

Taking the double time derivative of (2.10) yields

\[
\ddot{G}_{k,j,i} - \ddot{G}_{i,j,k} + s^2 \partial_i^4 G_{ij} = 0.
\]

On the other hand, taking the divergence of (2.10) (or, equivalently, summing up the \( i \)th derivative of (2.10) over \( i \)) gives

\[
(s^2)_{,i} \ddot{G}_{ij} + s^2 \dddot{G}_{ij,i} = 0.
\]

To write (2.38) in terms of the generalized functions \( f_+^{(k)} \), we compute the following spatial derivatives of \( G \) using Proposition 2.2:

\[
\ddot{G}_{k,j,k} = (B^l_{kj} f_+^{(-3+l)}(t^2 - \mathcal{T}))_{,k} = B^l_{kj} f_+^{(-3+l)}(t^2 - \mathcal{T}) - B^l_{kj} f_+^{(-3+l)'}(t^2 - \mathcal{T}) \mathcal{T},
\]

\[
= B^l_{kj} f_+^{(-3+l)}(t^2 - \mathcal{T}) - B^l_{kj} \mathcal{T} f_+^{(-4+l)}(t^2 - \mathcal{T}),
\]

\[
\dddot{G}_{k,j} = \mathcal{T} f_+^{(-4+l)}(t^2 - \mathcal{T}),
\]

\[
\dddot{G}_{i,j} = \mathcal{T} f_+^{(-4+l)}(t^2 - \mathcal{T}),
\]

where we enforce all dyadic coefficients \( B^l = O(1) \) as \( r \to 0^+ \).
\[ \tilde{G}_{kj,ki} = (B^l_{kj,k}f^{(-3+l)}_+(t^2 - \mathcal{T}) - B^l_{kj,k}T_+ f^{(-4+l)}_+(t^2 - \mathcal{T}))_i + B^l_{kj} T_+ f^{(-5+l)}_+(t^2 - \mathcal{T}), \]

and

\[ \tilde{G}_{ij, kk} = B^l_{ij, kk}f^{(-3+l)}_+(t^2 - \mathcal{T}) - [B^l_{ij, k}T_+ f^{(-4+l)}_+(t^2 - \mathcal{T}) + B^l_{ij} T_+ f^{(-5+l)}_+(t^2 - \mathcal{T}), \]

and the following double time derivative of \( \tilde{G} \):

\[
\partial_t^2 G_{ij} = B^l_{ij} \partial_t^2 f^{(-3+l)}_+(t^2 - \mathcal{T})
= B^l_{ij} \partial_t [2t f^{(-4+l)}_+(t^2 - \mathcal{T})]
= 2B^l_{ij} f^{(-4+l)}_+(t^2 - \mathcal{T}) + 4t^2 B^l_{ij} f^{(-5+l)}_+(t^2 - \mathcal{T})
= [2 + 4(l - 4)]B^l_{ij} f^{(-4+l)}_+(t^2 - \mathcal{T}) + 4t^2 B^l_{ij} f^{(-5+l)}_+(t^2 - \mathcal{T}).
\]

Inserting the above derivatives into (2.38) and (2.39) yields

\[
0 = \sum_{l=0}^{\infty} (B^l_{kj,ki} - B^l_{ij, kk})f^{(-3+l)}_+(t^2 - \mathcal{T}) + [(2 + 4(l - 4))s^2 B^l_{ij} - [B^l_{kj, k}T_+ + (B^l_{kj} T_+), i] + B^l_{kj} T_+ f^{(-4+l)}_+(t^2 - \mathcal{T}) + [B^l_{ij, k} T_+ - B^l_{ij} T_+ f^{(-4+l)}_+(t^2 - \mathcal{T}) + 4t^2 B^l_{ij} s^2 f^{(-5+l)}_+(t^2 - \mathcal{T}) \right. \\
= \sum_{l=-\infty}^{\infty} \left[ (B^l_{kj, ki} - B^l_{ij, kk}) + [2 + 4(l - 3)]s^2 B^l_{ij, i} \\
- [B^{l+1}_{kj, k}T_+ + (B^{l+1}_{kj} T_+), i] + [B^{l+1}_{ij, k} T_+ + (B^{l+1}_{ij} T_+), k] + B^{l+2}_{kj} T_+ T_+ f^{(-3+l)}_+(t^2 - \mathcal{T}) + 4t^2 B^{l+2}_{ij} s^2 f^{(-4+l)}_+(t^2 - \mathcal{T}) \right]
\]

and

\[
0 = \sum_{l=0}^{\infty} (s^2)_l B^l_{ij} f^{(-3+l)}_+ + s^2 B^l_{ij, i} f^{(-3+l)}_+(t^2 - \mathcal{T}) - s^2 B^l_{ij} T_+ f^{(-4+l)}_+(t^2 - \mathcal{T}) \\
= \sum_{l=-\infty}^{\infty} \{ (s^2 B^l_{ij, i} - s^2 B^{l+1}_{ij} T_+) f^{(-3+l)}_+(t^2 - \mathcal{T}) \},
\]

where we have made the convention that \( B^l \equiv 0 \) for \( l = -1, -2, \ldots \). In either of the above two equations, the coefficient of \( f^{(-3+l)}_+ \) should be equated to zero since, in an asymptotic series whose sum is asymptotically zero, each term must separately be zero. Consequently, we obtain the following two equations:
\[ 0 = (B_{ij}^{l+1})_{kk} - B_{ij}^{l} + [2 + 4(l - 3)s^2]B_{ij}^{l+1} + [B_{ij}^{l+1}_k, k] + (B_{ij}^{l+1})_{kk}, k] \]
\[ + B_{ij}^{l+2}T_{ij}^0 + B_{ij}^{l+2}T_{ij}^k + 4\mathcal{J}B_{ij}^{l+2}s^2, \]
\[ (2.40) \]
\[ 0 = (s^2 B_{ij}^{l})_{ik} - s^2 B_{ij}^{l+1}T_{ij}, i \]
\[ (2.41) \]
for \( l \in \mathbb{Z} \). Equation (2.41) further simplifies (2.40) into
\[ 0 = (B_{ij}^{l})_{kk} - B_{ij}^{l} + [2 + 4(l - 3)s^2]B_{ij}^{l+1} + [B_{ij}^{l+1}_k, k] + (B_{ij}^{l+1})_{kk}, k] \]
\[ + s^{-2}B_{ij}^{l+1}T_{ij}^0 + 4\mathcal{J}s^2 - T_{ij}^k + B_{ij}^{l+2}. \]
\[ (2.42) \]

Now we are ready to discuss the governing equations of \( \mathbf{B}^l \) and \( \mathcal{J} \).

### 2.2.1. Governing equation for \( \mathcal{J} \)
Setting \( l = -2 \) in (2.42), we obtain
\[ \mathcal{J}_{ij}^0 \mathcal{J}_{ij}^0 = 4s^2\mathcal{J}B_{ij}^{0}. \]
\[ (2.43) \]

It is clear that \( B_{ij}^{0} \) is nonzero since otherwise Hadamard’s ansatz (2.32) cannot recover the Green’s function \( \mathbf{G} \) in a homogeneous medium. Consequently, we obtain the governing equation for \( \mathcal{J} \),
\[ \mathcal{J}_{ij} \mathcal{J}_{ij} = 4s^2\mathcal{J}, \]
\[ (2.44) \]
or, in vector notation,
\[ |\nabla\mathcal{J}|^2 = 4s^2\mathcal{J}. \]
\[ (2.45) \]
This further indicates that \( \mathcal{J} \) is a nonnegative function in the sense that \( \mathcal{J} = \tau^2 \) for some nonnegative function \( \tau(\mathbf{r}; \mathbf{r}_0) \). To make sure that \( \tau = s_0 \tau \) for a homogeneous medium with \( s(\mathbf{r}) \equiv s_0 \), \( \tau \) is in general a nonzero function. From (2.45), we may further obtain the governing equation for \( \tau \),
\[ |\nabla\tau|^2 = s^2, \]
\[ (2.46) \]
a.k.a. the eikonal equation.

### 2.2.2. Governing equation for \( \mathbf{B}^l \)
Now by (2.44), equation (2.42) reduces to
\[ 2B_{ij}^{l+1}T_{ij} + [T_{kk} + 2 + 4(l - 3)s^2]B_{ij}^{l+1} \]
\[ = -s^{-2}s^2B_{kij}^{l+1}T_{ij}^0 + B_{ij}^{l+1}_k + (s^{-2}s^2B_{kij}^{l+1})_k \]
\[ (2.47) \]
or, in vector notation but with \( l \) replaced by \( l - 1 \),
\[ (\nabla \cdot \mathcal{J}) \mathbf{B}^{l} + \left[ \frac{\Delta \mathcal{J}}{2} + (2l - 7)s^2 \right] \mathbf{B}^{l} = -\nabla \mathcal{J} \frac{\nabla s^2 \cdot \mathbf{B}^{l}}{2s^2} + \frac{\Delta \mathbf{B}^{l-1}}{2} + \nabla \left( \frac{\nabla s^2 \cdot \mathbf{B}^{l-1}}{2s^2} \right). \]
\[ (2.48) \]
Meanwhile, (2.41) becomes
\[ \nabla \mathcal{J} \cdot \mathbf{B}^{l} = \frac{\nabla \cdot (s^2 \mathbf{B}^{l-1})}{s^2}. \]
\[ (2.49) \]
2.2.3. Initial conditions for $\mathbf{J}$ and $\mathbf{B}$. Let us discuss the initial condition of $\mathbf{J}$ first. Clearly, $\mathbf{G}$ itself should be singular (or undefined) when $\mathbf{r} = \mathbf{r}_0$ and $t = 0$. This indicates that $\mathbf{J}(\mathbf{r}_0; \mathbf{r}_0) = 0$ so that $\tau(\mathbf{r}_0; \mathbf{r}_0) = 0$. Assuming that both $s^2$ and $\mathbf{J}$ are analytic at the source $\mathbf{r}_0$, it has been shown in [18, 25] that

\[
(2.50) \quad \left( \frac{\mathbf{J}}{s_0^2 \mathbf{r}^2} \right)^\alpha = 1 + \frac{\alpha}{2S_0} \left( \frac{S_1}{2} + \frac{S_2}{3} - \frac{|\nabla S_1|^2}{48S_0} \mathbf{r}^2 \right) + \frac{\alpha}{4} \left( \frac{\alpha}{2} - 1 \right) \frac{S_2^2}{4S_0^2} + O(r^3),
\]

where $S_k$ is the degree-$k$ term in the Taylor expansion of $s^2$ about $\mathbf{r}_0$ such that

\[
(2.51) \quad s^2(\mathbf{r}) = \sum_{k=0}^\infty S_k(\mathbf{r}; \mathbf{r}_0).
\]

Next, to derive initial conditions for $\mathbf{B}$, we need to study the relation of $\mathbf{G}$ and $\mathbf{G}^0$ at the source point $\mathbf{r}_0$ first, where $\mathbf{G}^0$ is the Green's function, with $ij$th entry (2.24) for $1 \leq i, j \leq 3$ in a homogeneous medium with $s \equiv s_0$. For convenience, we write again the governing equations of the two as follows:

\[
(2.52) \quad \nabla \times \nabla \times \mathbf{G} + s^2 \mathbf{G} = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t),
\]

\[
(2.53) \quad \nabla \times \nabla \times \mathbf{G}^0 + s_0^2 \mathbf{G}^0 = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}_0) \delta(t).
\]

Considering the difference between (2.52) and (2.53), we obtain

\[
(2.54) \quad \nabla \times \nabla \times (\mathbf{G} - \mathbf{G}^0) + s^2(\mathbf{G} - \mathbf{G}^0) = (s_0^2 - s^2)\mathbf{G}^0.
\]

In subscript notation, it becomes

\[
\begin{align*}
(G_{kl} - G^0_{kl})_{,kl} - (G_{ij} - G^0_{ij})_{,kk} + s^2(G_{ij} - \tilde{G}^0_{ij}) \\
= (s_0^2 - s^2)\tilde{G}_{ij} \\
= (s_0^2 - s^2)\left[ \frac{\delta_{ij} - 3\gamma_i\gamma_j}{4\pi s_0^2 r^4} \right] \delta(t - s_0 \mathbf{r}) + \frac{s_0^2 r \delta(1)}{s_0^2 s_0^2 r} \delta(t - s_0 \mathbf{r}) \\
+ \frac{(s_0^2 - s^2) \delta_{ij} - \gamma_i \gamma_j}{4\pi r} \delta(2)(t - s_0 \mathbf{r})
\end{align*}
\]

(2.55)

Taking the double time derivative yields

\[
\begin{align*}
(G_{kl} - G^0_{kl})_{,kl} - (G_{ij} - G^0_{ij})_{,kk} + s^2 \partial_t^2(\tilde{G}_{ij} - \tilde{G}^0_{ij}) \\
= (s_0^2 - s^2)\left[ \frac{\delta_{ij} - 3\gamma_i\gamma_j}{4\pi s_0^2 r^3} \right] \delta(t - s_0 \mathbf{r}) + \frac{s_0^2 r \delta(1)}{s_0^2 s_0^2 r} \delta(t - s_0 \mathbf{r}) \\
+ \frac{(s_0^2 - s^2) \delta_{ij} - \gamma_i \gamma_j}{4\pi r} \delta(4)(t - s_0 \mathbf{r})
\end{align*}
\]

(2.56)

For any function $g \in C^\infty_0(\mathbb{R})$ with compact support $[0, T]$ for some $T > 0$, integrating $g$ times the right-hand side of (2.56) over $[0, T]$ yields

\[
\begin{align*}
\int_0^T \left\{ (s_0^2 - s^2)\left[ \frac{\delta_{ij} - 3\gamma_i\gamma_j}{4\pi s_0^2 r^3} \right] \delta(t - s_0 \mathbf{r}) + \frac{s_0^2 r \delta(1)}{s_0^2 s_0^2 r} \delta(t - s_0 \mathbf{r}) \\
+ \frac{(s_0^2 - s^2) \delta_{ij} - \gamma_i \gamma_j}{4\pi r} \delta(4)(t - s_0 \mathbf{r}) \right\} g(t) dt \\
= (s_0^2 - s^2)\left[ \frac{\delta_{ij} - 3\gamma_i\gamma_j}{4\pi s_0^2 r^3} \right] g''(s_0 \mathbf{r}) - s_0 r g^{(3)}(s_0 \mathbf{r}) + (s_0^2 - s^2)\left( \frac{\delta_{ij} - \gamma_i \gamma_j}{4\pi r} \right) g^{(4)}(s_0 \mathbf{r})
\end{align*}
\]

(2.57)
Hadamard’s ansatz (2.31) and the closed form (2.25) of $\dddot{r}$ as $T \to 0^+$.

Therefore, we have to enforce that as $r \to 0^+$,

$$0 \left( \frac{1}{r^2} \right) = \int_0^T \left\{ (\dddot{G}_{kj} - \dddot{G}_{kij}),_{ki} - (\dddot{G}_{ij} - \dddot{G}_{ij0}) ,_{kk} + s^2 \dddot{\gamma}^2 (\dddot{G}_{ij} - \dddot{G}_{ij0}) \right\} g(t) dt$$

$$= \int_0^T \left\{ [ (\dddot{G}_{kj} - \dddot{G}_{kij})g(t)]_{,ki} - [ (\dddot{G}_{ij} - \dddot{G}_{ij0})g(t)]_{,kk} \right\} dt$$

$$+ s^2 \int_0^T (\dddot{G}_{ij} - \dddot{G}_{ij0}) \dddot{g}(t) dt$$

$$= \left\{ \left[ \int_0^T (\dddot{G}_{kj} - \dddot{G}_{kij})g(t) \right]_{,ki} - \left[ \int_0^T (\dddot{G}_{ij} - \dddot{G}_{ij0})g(t) \right]_{,kk} \right\}$$

$$+ s^2 \int_0^T (\dddot{G}_{ij} - \dddot{G}_{ij0}) \dddot{g}(t) dt.$$ (2.58)

This motivates us to propose the following matching condition:

$$\int_0^T (\dddot{G}_{ij}(r, t; r_0) - \dddot{G}_{ij0}(r, t; r_0)) g(t) dt = O(1)$$ (2.59)

as $r \to 0^+$. Otherwise, if (2.59) is not satisfied, then the first term on the right-hand side of (2.58) will make the singularity at $r = 0$ worse than $O(1)$.

As any function $g \in C^\infty_c([0, T])$ can be uniformly approximated by a polynomial function due to the Weierstrass approximation theorem, the matching condition (2.59) in fact can be equivalently restated as follows: for any nonnegative integer $m$ and any $T > 0$,

$$\int_0^T (\dddot{G}(r, t; r_0) - \dddot{G}^0(r, t; r_0)) t^m dt = O(1)$$ (2.60)

as $r \to 0^+$.

Based on (2.60), we are ready to derive initial conditions for $B^i$. Inserting Hadamard’s ansatz (2.31) and the closed form (2.25) of $\dddot{G}^0$ into (2.60) yields

$$B^0 \int_0^T \delta^{(2)}(t^2 - \mathcal{T}) t^m dt + B^1 \int_0^T \delta^{(1)}(t^2 - \mathcal{T}) t^m dt + B^2 \int_0^T \delta(t^2 - \mathcal{T}) t^m dt$$

$$- \frac{2s_0^3}{\pi} (I - \tilde{g}^T) \int_0^T \delta^{(2)}(t^2 - s_0^2r^2) t^m dt + \frac{2s_0}{\pi} \int_0^T \delta^{(1)}(t^2 - s_0^2r^2) t^m dt$$

$$+ \sum_{i=0}^{\infty} B^{i+3} \int_0^T f_+^{(i)}(t^2 - \mathcal{T}) t^m dt = O(1).$$ (2.61)

As $f_+^{(i)}$ are bounded on $[0, T]$ for $i = 0, 1, \ldots$, $\int_0^T f_+^{(i)}(t^2 - \mathcal{T}) t^m dt = O(1)$. Getting rid of all $O(1)$ terms in (2.61), we get
To further simplify (2.62), we need the following formula:

\[
\mathbf{B}^0 \int_0^T \delta(\iota^2 - \mathcal{T}) \iota^m \, d\iota + \mathbf{B}^1 \int_0^T \delta^{(1)}(\iota^2 - \mathcal{T}) \iota^m \, d\iota + \mathbf{B}^2 \int_0^T \delta(\iota^2 - \mathcal{T}) \iota^m \, d\iota
\]

(2.62) \quad \frac{2s_0^3r^2}{\pi} \int_0^T \delta(\iota^2 - \tilde{\mathcal{T}}^2) \iota^m \, d\iota + \frac{2s_0^3}{\pi} \int_0^T \delta^{(1)}(\iota^2 - s_0^2r^2) \iota^m \, d\iota = O(1).

To further simplify (2.62), we need the following formula:

\[
\int_0^T \delta(\iota^2 - \mathcal{T}) \iota^m \, d\iota = \int_{-\tau}^{T-\tau} \delta^{(k)}(\iota) \frac{1}{2} (\bar{i} + \mathcal{T}(x))^{m-\frac{1}{2}} \, d\bar{i}
\]

\[
= \frac{(-1)^k}{2} \left[ (\bar{i} + \mathcal{T})^{m-\frac{1}{2}} \right]^{(k)} \bigg|_{\bar{i}=0}
\]

\[
= \frac{(-1)^k k!}{2} \prod_{j=0}^{k-1} \left( \frac{m}{2} - \frac{1}{2} - j \right) \mathcal{T}^{m-\frac{1}{2} - k}.
\]

(2.63)

For \( k \leq 2 \), the integral in (2.63) is \( O(1) \) when \( m \geq 5 \) since \( \frac{m}{2} - \frac{1}{2} - k \geq 0 \). Thus, considering only \( m = 0, 1, 2, 3, 4 \) respectively in (2.62) and using (2.63) with \( k \leq 2 \), we obtain as \( r \to 0^+ \) that

\[
\frac{3}{8} \left[ \mathbf{B}^0 - \frac{2s_0^3r^2}{\pi(s_0r)^3} (\mathbf{I} - \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T) \right] + \frac{1}{4} \left[ \mathbf{B}^1 + \frac{2s_0 I}{\pi(s_0r)^3} \right] + \frac{\mathbf{B}^2}{2\mathcal{T}} = O(1),
\]

(2.64)

\[
\frac{\mathbf{B}^2}{2} = O(1),
\]

\[
- \frac{1}{8} \left[ \mathbf{B}^0 - \frac{2s_0^3r^2}{\pi(s_0r)^3} (\mathbf{I} - \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T) \right] - \frac{1}{4} \left[ \mathbf{B}^1 + \frac{2s_0 I}{\pi(s_0r)^3} \right] + \frac{\mathbf{B}^2}{2\mathcal{T}} = O(1),
\]

(2.66)

\[
- \frac{1}{2} \left[ \mathbf{B}^1 + \frac{2s_0}{\pi I} \right] + \frac{1}{2} \mathbf{B}^2 \mathcal{T} = O(1),
\]

(2.67)

\[
\frac{3}{8} \left[ \mathbf{B}^0 - \frac{2s_0^3r^2}{\pi s_0 r} (\mathbf{I} - \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T) \right] + \frac{3}{4} \left[ \mathbf{B}^1 \mathcal{T} + \frac{2s_0^3r}{\pi} \right] + \frac{1}{2} \mathbf{B}^2 \mathcal{T}^2 = O(1),
\]

(2.68)

Hiding all \( O(1) \) terms gives rise to

\[
\frac{3}{8} \left[ \mathbf{B}^0 + \frac{1}{4} \mathbf{B}^1 \right] + \frac{\mathbf{B}^2}{2\mathcal{T}} - \frac{\mathbf{I} - \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T}{4\pi r} = O(1),
\]

(2.69)

\[
\frac{1}{8} \frac{\mathbf{B}^0}{\mathcal{T}} + \frac{1}{4} \frac{\mathbf{B}^1}{\mathcal{T}} + \frac{\mathbf{B}^2}{4\mathcal{T}^2} - \frac{\mathbf{I} - \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T}{4\pi r} = O(1),
\]

(2.70)
Finally, (2.69) implies (2.71) due to (2.50). Next, (2.69) leads to an initial condition for $B^1$:

$$
B^1 = -\frac{3B^0}{2T} + \frac{\mathcal{J}^2}{\pi s^2 r^3} (I - 3\hat{r}\hat{r}^T) - 2B^2\mathcal{J} + O(\mathcal{J}^\frac{3}{2})
$$

(2.74)

Finally, (2.69) gives an initial condition for $B^2$:

$$
B^2 = -\frac{3B^0}{4T^2} - \frac{B^1}{2T} + \frac{\mathcal{J}^2}{2\pi s^2 r^3} (I - 3\hat{r}\hat{r}^T) + O(\mathcal{J}^\frac{3}{2}).
$$

(2.75)

Summarizing all the previous results, we have obtained the following theorem.

**Theorem 2.1.** Suppose a dyadic function $G(r, t; r_0)$ of the form (2.31) solves the time-domain Maxwell’s equations (2.1) and (2.2). The following statements hold:

(a) The nonzero function $\mathcal{J}$ is governed by

$$
|\nabla \mathcal{J}|^2 = 4s^2\mathcal{J},
$$

with the initial condition $\mathcal{J}(r_0; r_0) = 0$.

(b) The dyadic coefficients $\{B^l\}_{l=0}^\infty$ are governed by the following recursive equations:

$$
(\nabla \mathcal{J} \cdot \nabla)B^l + \left[\frac{\Delta \mathcal{J}}{2} + (2l - 7)s^2 \right] B^l = -\nabla \mathcal{J} \frac{s^2 \cdot B^l}{2s^2} + \frac{\Delta B^{l-1}}{2}
$$

$$
+ \nabla \left( \frac{\nabla s^2 \cdot B^{l-1}}{2s^2} \right),
$$

(2.77)

$$
\nabla \mathcal{J} \cdot B^l = \frac{\nabla \cdot (s^2B^{l-1})}{s^2}.
$$

(2.78)

To fulfill (2.60), $B^0$, $B^1$, and $B^2$ asymptotically behave near $r_0$ as follows:

$$
B^0 = \frac{\mathcal{J}^2}{\pi s^2 r^3} (I - 3\hat{r}\hat{r}^T) + \frac{\mathcal{J}^2}{\pi r} (I + \hat{r}\hat{r}^T) + O(\mathcal{J}^\frac{3}{2}),
$$

(2.79)

$$
B^1 = -\frac{3B^0}{2T} + \frac{\mathcal{J}^2}{\pi s^2 r^3} (I - 3\hat{r}\hat{r}^T) + O(\mathcal{J}),
$$

(2.80)

$$
B^2 = -\frac{3B^0}{4T^2} - \frac{B^1}{2T} + \frac{\mathcal{J}^2}{2\pi s^2 r^3} (I - 3\hat{r}\hat{r}^T) + O(\mathcal{J}^\frac{3}{2})
$$

as $r \to 0^+$. 

2.3. Hadamard–Babich ansatz in the frequency domain. The Fourier transform of (2.1) in time yields the FDPS Maxwell’s equations (1.1) with

\[ G(r; r_0) = \int_0^\infty G(r, t; r_0) e^{i\omega t} dt, \]
so that, by (2.31), we get the following frequency-domain asymptotic ansatz:

\[ G(r; r_0) = \sum_{l=0}^\infty B_l(r; r_0) - \omega^2 \int_0^\infty e^{i\omega t} f_l^\prime \left( t^2 - r^2 (r; r_0) \right) dt. \]

The integral in (2.83) has the following closed form.

**Proposition 2.3.**

\[ \int_\tau^\infty e^{i\omega t} f \left( \nu - \frac{1}{2}, t^2 \right) dt = \frac{1}{\sqrt{\pi}} \left( \frac{2\pi}{\omega} \right)^\nu e^{i\pi\nu} H_{1/2}^{(1)}(\omega \tau) \]

Here \( f_\nu(\omega, \tau) \) is exactly the basis function used in Babich’s ansatz [2].

**Proof.** Magnus and Oberhettinger [19, p. 28] give the following integral representation of the Hankel functions:

\[ H_{\nu}^{(1,2)}(x) = \frac{\Gamma \left( \frac{1}{2} - \nu \right) \left( \frac{x}{2} \right)^\nu}{\pm i\pi \Gamma \left( \frac{1}{2} \right)} \left[ 1 - e^{\mp 2i\pi(\nu - \frac{1}{2})} \right] \int_1^\infty e^{\pm ixt} \left( t^2 - 1 \right)^{\nu - \frac{1}{2}} dt. \]

Here \( -\frac{1}{2} < \text{Re} \nu < \frac{1}{2} \); the upper signs are for \( H_{\nu}^{(1)} \) and the lower signs are for \( H_{\nu}^{(2)} \).

Notice that

\[ 1 - e^{\mp 2i\pi(\nu - \frac{1}{2})} = 1 + e^{\mp 2i\pi\nu} = e^{\mp i\pi\nu} \left[ e^{\pm i\pi\nu} + e^{\mp i\pi\nu} \right] = 2e^{\mp i\pi\nu} \cos \pi\nu. \]

Abramowitz and Stegun [1, Formula 6.1.17] give the reflection formula for the \( \Gamma \) function:

\[ \Gamma(z)\Gamma(1 - z) = \pi \csc \pi z. \]

In (2.87) set \( z = \nu + \frac{1}{2} \) to get

\[ \Gamma \left( \nu + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} - \nu \right) = \pi \csc \pi \left( \nu + \frac{1}{2} \right) = \frac{\pi}{\cos \pi\nu}. \]

So

\[ \frac{\cos \pi\nu \Gamma \left( \frac{1}{2} - \nu \right)}{\pi} = \frac{1}{\Gamma(\nu + \frac{1}{2})} = \frac{1}{(\nu - \frac{1}{2})!}. \]

Using (2.86) and (2.89) in (2.85), we get

\[ H_{\nu}^{(1,2)}(x) = 2 \left( \frac{x}{2} \right)^\nu e^{\mp i\pi\nu} \int_1^\infty e^{\pm ixt} \left( t^2 - 1 \right)^{\nu - \frac{1}{2}} (\nu - \frac{1}{2})! dt, \]
and we have

\begin{equation}
\int_1^\infty e^{\pm i\pi x} f_+^{\nu-\frac{1}{2}} (t^2 - 1) dt = \pm \frac{1}{2} i \sqrt{\pi} \left( \frac{2}{x} \right)^{\nu} e^{\pm i \pi \nu} H^{(1,2)}_\nu(x)
\end{equation}

when \(-\frac{1}{2} < \Re \nu < \frac{1}{2}\), and for other values of \(\nu\) by analytic continuation. Setting \(t = \frac{t'}{\tau}\), \(dt = dt'/\tau\), we see that

\begin{equation}
\int_1^\infty e^{\pm i\pi x} f_+^{\nu-\frac{1}{2}} (t'^2 - \tau^2) dt' = \pm \frac{1}{2} i \sqrt{\pi} \left( \frac{2}{x} \right)^{\nu} e^{\pm i \pi \nu} H^{(1,2)}_\nu(x).
\end{equation}

Setting \(x = \omega \tau\), dropping the primes on \(t'\), and taking the upper sign, we get (2.84).

By (2.82) and Proposition 2.3, we immediately find that the frequency-domain Hadamard’s ansatz should be

\begin{equation}
G(r; r_0) = \sum_{l=0}^{\infty} \frac{B^l(r; r_0)}{-\omega^2} f_{-3+l+\frac{1}{2}}(\omega, \tau(r; r_0)),
\end{equation}

which we refer to as the Hadamard–Babich ansatz in the following.

Since

\begin{align}
f_{-3+l+\frac{1}{2}}(\omega, \tau(r; r_0)) &= \frac{i \sqrt{\pi}}{2} \left( \frac{2\tau}{\omega} \right)^{-3+l+\frac{1}{2}} e^{i\pi(-3+l+\frac{1}{2})} H^{(1)}_{-3+l+\frac{1}{2}}(\omega \tau) \\
&= \sqrt{\pi} \left( \frac{2 \tilde{\tau}}{k_0} \right)^{-3+l+\frac{1}{2}} (-1)^l H^{(1)}_{-3+l+\frac{1}{2}}(k_0 \tilde{\tau})
\end{align}

(2.94)

where \(\tilde{\tau} = \epsilon \tau\), we have

\begin{align}
G(r; r_0) &= \sum_{l=0}^{\infty} \left\{ \frac{(-2)^{l-3}}{\epsilon^{2l-3}} - B^l(r; r_0) \right\} \left\{ \frac{-\tilde{\tau}^{l-2} h_{l-3}^{(1)}(k_0 \tilde{\tau})}{k_0^{l-1}} \right\} \\
&= \sum_{l=0}^{\infty} \tilde{B}^l(r; r_0) \tilde{f}^l(\tilde{\tau}; k_0) \\
&= \sum_{l=0}^{\infty} \frac{A^l(r; r_0)}{\tilde{\tau}^{2l}} \tilde{f}^l(\tilde{\tau}; k_0)
\end{align}

(2.95)

where we have defined

\begin{equation}
\tilde{f}^l(\tilde{\tau}; k_0) = \frac{-\tilde{\tau}^{l-2} h_{l-3}^{(1)}(k_0 \tilde{\tau})}{k_0^{l-1}},
\end{equation}

(2.96)

\begin{equation}
\tilde{B}^l(r; r_0) = \frac{(-2)^{l-3}}{\epsilon^{2l-3}} B^l(r; r_0),
\end{equation}

(2.97)

\begin{equation}
A^l(r; r_0) = \tilde{B}^l \tilde{\tau}^{2l}.
\end{equation}

(2.98)

Since \(B^l = O(1)\) as \(r \to 0\), \(\tilde{B}^l = O(1)\) so that \(A^l = O(\tau^{2l})\). We introduce \(A^l\) here, since \(B^l\) may not be smooth at the source \(r_0\) for an inhomogeneous medium; this
can be seen from (2.50) and from the initial conditions (2.79)–(2.81). The right-hand side of (2.95) is exactly our Babich-like ansatz (1.3), so that we have established the equivalence of the Babich-like ansatz (1.3) and the Hadamard–Babich ansatz (2.93). For this reason, we will not differentiate (2.93) and (2.95) (or (1.3)), and we will call both the Hadamard–Babich ansatz from now on. We will also refer to the unknowns $A^l$ and $\tau$ as the Hadamard–Babich ingredients.

Noticing that $\mathcal{I} = \tau^2 = \frac{l^2}{2}$, we now can rewrite Theorem 2.1 in the framework of the Hadamard–Babich ansatz (2.95).

**Theorem 2.2.** Suppose a dyadic function $G(r; r_0)$ of the form (2.95) solves the FDPS Maxwell’s equations (1.1). The following statements hold:

(a) The nonzero function $\tau$ is governed by

$$|\nabla \tau|^2 = n^2,$$

with the initial condition $\tau(r_0; r_0) = 0$.

(b) The dyadic coefficients $\{A^l\}^\infty_{l=0}$ are governed by the following recursive equations:

$$\nabla \tau^2 \cdot A^l + \frac{\partial^2 \tau^2}{2} - (2l + 7)\tau^2 n^2 \right) A^l = -\nabla \tau^2 \cdot \nabla n^2 \cdot A^l + 2\nabla^2 \tau^2 \cdot A^l,$$

where we define for any $l \geq 0$,

$$R_{l-1} = -\tau^2 \nabla \left( \frac{\nabla n^2 \cdot A^{l-1}}{n^2} \right) + \frac{(l - 1)\nabla n^2 \cdot A^{l-1}}{n^2} \nabla \tau^2$$

$$+ (l - 1)(2\nabla A^{l-1} \cdot \nabla \tau^2 + A^{l-1} \Delta \tau^2),$$

and in particular $R_{-1} \equiv 0$. To fulfill the matching condition (2.60), $A^0$, $A^1$, and $A^2$ behave asymptotically near $r_0$ as follows:

$$A^0 = \frac{(I - 3\tau \tau^T)n_0 \tau^2 + (I + \tau \tau^T)n_0^3 r^2}{8\pi} \left( \frac{\tau}{n_0 r} \right)^3 + O(\tau^3),$$

$$A^1 = 3A^0 - \frac{1}{4\pi} \left( \frac{\tau}{n_0 r} \right)^3 (I - 3\tau \tau^T)\tau^2 n_0 + O(\tau^4),$$

$$A^2 = 3A^0 - A^1 - \frac{1}{4\pi} \left( \frac{\tau}{n_0 r} \right)^3 (I - 3\tau \tau^T)\tau^2 n_0 + O(\tau^5),$$

where $n_0 = n(r_0)$ is the refractive index at the source $r_0$.

With no effort, one sees immediately that Theorem 2.2 is the same as Theorem 2.6 in [15] only in different notations. However, they are now completely derived from the TDPS Maxwell’s equations (2.1) using Hadamard’s method.

To conclude this section, we remark that although $A^l$ are governed by (2.100) and (2.101), the former implies the latter. As was shown, (2.101) or (2.78) is originated from (2.39), which is just the divergence of (2.10) that gives (2.100). Alternatively, one can prove (2.101) by taking the dot product of (2.100) and $\nabla \tau^2$ and by induction on $l$; since the proof is tedious, we omit it here.
3. Huygens sweeping method. We may develop two different approaches to use the proposed Hadamard–Babich ansatz: one is an ordinary differential equation (ODE)–based Lagrangian approach, and the other is a partial differential equation (PDE)–based Eulerian approach. By a Lagrangian approach, we mean that we solve a set of ODEs along rays to compute the Hadamard–Babich ingredients so as to construct asymptotic wavefields; certainly, different strategies will be employed according to where caustics occur along a ray path, and we will develop such Lagrangian approaches to implement the ansatz in a future work.

Meanwhile, we remark that in [15] we have developed an Eulerian approach to implement the Hadamard–Babich ansatz (2.95) by assuming that no caustic occurs in a region of space containing a point source. However, as proved in [31], caustics occur with high probability in a generic inhomogeneous medium. Therefore, inspired by the work [22], assuming that the contributing rays emitted from a point source have a consistent orientation so that waves propagate close to a single direction, we develop a fast Huygens sweeping method to deal with caustics by incorporating the Hadamard–Babich ansatz (2.95) into the Huygens secondary-source principle.

As explained and illustrated in [16, 17, 22, 23], given a primary source point, we a priori partition the computational domain into several layers as sketched in Figure 1, where \( r_0 \) denotes the primary source, \( S_m \) is the \( m \)th secondary source plane \( z = z_m \), so that \( \Omega_{m+1} = \{(x, y, z)^T | z_m < z < z_{m+1}\} \), the wavefield associated with the primary source \( r_0 \) has no caustics in the region \( \Omega_0 \) but may develop caustics beyond \( \Omega_0 \), and, for all \( m \), the wavefields associated with the secondary sources situated on \( S_m \) have no caustics in \( \Omega_{m+1} \) but may develop caustics beyond \( \Omega_{m+1} \).

In \( \Omega_0 \), the Hadamard–Babich ansatz (2.95) is applicable for computing \( G(r; r_0) \) since no caustics occur. Numerically, it is impossible to compute all dyadic amplitudes \( A_l \) for \( l = 0, 1, 2, \ldots \). Following [15], we approximate \( G(r; r_0) \) by the first two terms of the Hadamard–Babich ansatz (2.95), i.e.,

\[
G(r; r_0) \approx A^0(r; r_0)f^0(\tilde{t}(r; r_0); k_0) + \frac{A^1(r; r_0)}{\tilde{t}^2(r; r_0)}f^1(\tilde{t}(r; r_0); k_0).
\]

(3.1)

To compute (3.1) numerically, we need to solve the eikonal equation (2.99) to obtain \( \tilde{t} \) and the vectorial Hamilton–Jacobi equations (2.100) and (2.101) for \( l = 0 \) and \( l = 1 \) to obtain \( A^0 \) and \( A^1 \), respectively. To obtain a first-order accurate \( G \), we need a fifth-order accurate \( \tilde{t} \), a third-order accurate \( A^0 \), and a first-order accurate \( A^1 \) since \( A^1 \) depends on \( \Delta \tilde{t}^2 \) and \( \Delta A^0 \). For the scalar eikonal equation (2.99), we employ the Lax–Friedrichs weighted essentially nonoscillatory (LxF-WENO)–based schemes [13, 17, 18, 23, 25, 28, 33, 34] to compute \( \tilde{t} \). For the vectorial equations

\[
\begin{align*}
S_0 & \quad \Omega_0 \quad r_0 \\
S_1 & \quad \Omega_1 \\
S_2 & \quad \Omega_2 \\
S_3 & \quad \Omega_3
\end{align*}
\]

Fig. 1. A two-dimensional sketch of Huygens sweeping method. The computational domain is partitioned into four layers. The primary source \( r_0 \). The secondary source plane \( S_m \) and its associated region \( \Omega_{m+1} \) for \( m = 0, 1, 2 \).
(2.100) and (2.101), since the components of $A^l$ are coupled together, we need to first use a decoupling approach in [15] to transform the vectorial equations into several scalar Hamilton–Jacobi equations, and we then solve those scalar equations by the LxF–WENO schemes to get $A^0$ and $A^1$; see [15] and the references therein for details. In passing, we remark that the high-order schemes for the eikonal and transport equations that we are using here were developed in [18, 23], which in turn are based on the Lax–Friedrichs sweeping [12, 13, 26, 27, 28, 30, 33, 34], WENO finite difference approximation [10, 11, 14, 20], and factorization of the upwind source singularities [6, 18, 21, 32]. To treat the upwind singularity at the point source, an adaptive method for the eikonal and transport equations has been proposed in [24] as well.

Beyond $\Omega_0$, we construct $G(r; r_0)$ by using the Huygens–Kirchhoff formula in each $\Omega_{m+1}$ in a layer-by-layer manner. Suppose now $G(r; r_0)$ becomes available for $r \in \Omega_m$ so that $G(r; r_0)$ on $S_m$ is known. As shown in [4, 22], for any $r' \in \Omega_{m+1}$, the Huygens–Kirchhoff formula gives

\begin{equation}
G_j(r'; r_0) = -\int_{S_m} \nu(r) \cdot [G_j(r; r_0) \times \nabla \times G(r; r') + \nabla \times G_j(r; r_0) \times G(r; r')] \, dS(r),
\end{equation}

where $\nu(r)$ denotes the unit normal vector at $r$ on $S_m$ and $G_j$ is the $j$th column of $G$ for $j = 1, 2, 3$. To make use of (3.2), we need to compute $G(r; r')$ and its curl $\nabla \times G(r; r')$, where the associated source $r'$ is in $\Omega_{m+1}$. Directly using the two-term ansatz (3.1) to compute these two terms is costly, as we need to solve the governing equations (2.99)–(2.101) for each source point $r'$ in the three-dimensional manifold $\Omega_{m+1}$, at $r$ on only a two-dimensional manifold $S_m$. To reduce the cost, we use the reciprocal relation to interchange the roles of $r$ and $r'$ so that only the secondary source points on the two-dimensional manifold $S_m$ are involved in the above computational process.

Specifically, since $G(r'; r)$ is assumed to have no caustics in $r' \in \Omega_{m+1}$, by the reciprocal relation and by (3.1) we may approximate

\begin{equation}
G(r; r') = G^T(r'; r) \approx A^{0,T}(r'; r) \tilde{f}^0(\tilde{\tau}(r'; r); k_0) + \frac{A^{1,T}(r'; r)}{\tilde{\tau}^2(r'; r)} \tilde{f}^1(\tilde{\tau}(r'; r); k_0),
\end{equation}

where the superscript $T$ indicates the transpose. Then, using the properties of $\tilde{f}^l$ listed in Lemma 2.2 in [15], the curl of $G(r; r')$ becomes

\begin{align}
\nabla \times G(r; r') & \approx [A^{0,T}(r'; r) \times \nabla_r \tilde{\tau}(r'; r)] \tilde{f}^0(\tilde{\tau}(r'; r); k_0) \\
& + \left[ \nabla_r \times A^{0,T}(r'; r) + \frac{A^{1,T}(r'; r)}{\tilde{\tau}(r'; r)} \times \nabla_r \tilde{\tau}(r'; r) \right] \tilde{f}^0(\tilde{\tau}(r'; r); k_0) \\
& + \nabla_r \times \frac{A^{1,T}(r'; r)}{\tilde{\tau}^2(r'; r)} \tilde{f}^1(\tilde{\tau}(r'; r); k_0),
\end{align}

where the subscript $r$ in $\nabla_r$ emphasizes that the gradient operator is applied with respect to $r$. Since $\tilde{f}^0(\tilde{\tau}(r'; r); k_0) = \mathcal{O}(k_0^{-1})$ as $k_0 \to \infty$, retaining only the leading-order terms, i.e., the $\mathcal{O}(1)$ terms in (3.3) and the $\mathcal{O}(k_0)$ terms in (3.4), we obtain
We can apply the LxF-WENO schemes to compute (3.9)
\[ G(\mathbf{r}; \mathbf{r}') \approx \frac{A^{0,T}(\mathbf{r}'; \mathbf{r})}{\tau^{3}(\mathbf{r}'; \mathbf{r})} e^{i k_0 \tau(\mathbf{r}'; \mathbf{r})}, \]
(3.6) \[ \nabla \times G(\mathbf{r}; \mathbf{r}') \approx \frac{i k_0}{\tau^{3}(\mathbf{r}'; \mathbf{r})} \left[ A^{0,T}(\mathbf{r}'; \mathbf{r}) \times \nabla r \tau(\mathbf{r}'; \mathbf{r}) \right] e^{i k_0 \tau(\mathbf{r}'; \mathbf{r})}. \]

Such approximations are adequately accurate for computing \( G_j(\mathbf{r}'; \mathbf{r}_0) \) in (3.2) for the following reason. The integrand in (3.2) is in general \( O(k_0) \) and becomes \( O(1) \) only when \( \nabla \times G(\mathbf{r}; \mathbf{r}') = O(1) \) and \( G(\mathbf{r}; \mathbf{r}') = O(1/k_0) \) (since \( \nabla \times G_j(\mathbf{r}; \mathbf{r}_0) = O(k_0) \)). Therefore, the approximations lose accuracy only when the right-hand sides of (3.5) and (3.6) are simultaneously equal to 0, and the points with such a property constitute only a set of measure 0 and will not affect the value of \( G_j(\mathbf{r}'; \mathbf{r}_0) \) in (3.2).

As can be seen, we need the following three unknown ingredients: \( A^0(\mathbf{r}'; \mathbf{r}) \), \( \tau(\mathbf{r}'; \mathbf{r}) \), and \( \nabla r \tau(\mathbf{r}'; \mathbf{r}) \) at each secondary source \( \mathbf{r} \in S_m \). Among these, only the governing equation of \( \nabla r \tau(\mathbf{r}'; \mathbf{r}) \) is unknown. According to [15],
\[ \nabla r \tau(\mathbf{r}'; \mathbf{r}) = t^{(0)}(\mathbf{r}'; \mathbf{r}) n(\mathbf{r}), \]
where \( t^{(0)} = (t_1^{(0)}, t_2^{(0)}, t_3^{(0)})^T \) denotes the take-off direction of the ray from \( \mathbf{r} \) to \( \mathbf{r}' \) and is governed by
\[ (\nabla r' \tau^2 \cdot \nabla r') t^{(0)}(\mathbf{r}'; \mathbf{r}) = 0 \]
with the initial condition
\[ \lim_{r' \to \mathbf{r}} \left[ t^{(0)}(\mathbf{r}'; \mathbf{r}) - \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|} \right] = 0. \]

We can apply the LxF-WENO schemes to compute \( t^{(0)} \). Consequently, using (3.5) and (3.6), equation (3.2) can be approximated by
\[ G_j(\mathbf{r}'; \mathbf{r}_0) \approx \int_{S_m} \left[ i k_0 n(\mathbf{r}) (\nu(\mathbf{r}) \cdot t^{(0)}(\mathbf{r}'; \mathbf{r})) (G_j(\mathbf{r}; \mathbf{r}_0) \cdot A^{0,T}(\mathbf{r}'; \mathbf{r})) \right. \]
\[ - i k_0 n(\mathbf{r}) (G_j(\mathbf{r}; \mathbf{r}_0) \cdot t^{(0)}(\mathbf{r}'; \mathbf{r})) (\nu(\mathbf{r}) \cdot A^{0,T}(\mathbf{r}'; \mathbf{r})) \]
\[ - (\nu(\mathbf{r}) \times \nabla \times G_j(\mathbf{r}; \mathbf{r}_0)) \cdot A^{0,T}(\mathbf{r}'; \mathbf{r}) \left. \frac{e^{i k_0 \tau(\mathbf{r}'; \mathbf{r})}}{\tau^{3}(\mathbf{r}'; \mathbf{r})} dS(\mathbf{r}) \right. \]
(3.10)
for \( \mathbf{r}' \in \Omega_{m+1}. \) Using (3.10) to sweep through all layers \( \{\Omega_m\} \), we can obtain \( G(\mathbf{r}; \mathbf{r}_0) \) in the whole computational domain. For implementation details of the above method, we refer readers to [22].

4. Numerical examples. In this section, we will study several numerical examples. Unless otherwise stated, all computations were executed on a 20-core 2.5-GHz Intel Xeon E5-2670v2 processor with 64 GB of RAM at the High Performance Computing Center at Michigan State University. To obtain a reference solution, if necessary, we apply the FDTD method [29] directly to the associated time-domain Maxwell’s equations (2.10)–(2.12) to obtain a numerical solution in the time domain and then the Fourier transform in time to compute a numerical solution in the frequency domain.
Example 1. Constant refractive index model. In this example, the Green’s function is constructed with the following setup:

- The refractive index function is \( n(x, y, z) \equiv 1 \).
- The computational domain is \( \Omega = [0, 2] \times [0, 2] \times [0, 2] \), and the two-term Hadamard–Babich ingredients are computed on a uniformly spacing grid with size 51 \( \times \) 51 \( \times \) 51.
- The primary source point is \( \mathbf{r}_0 = (1.0, 1.0, 0.2)^T \).
- One secondary source plane is placed at \( S_0 : z = 1.2 \), and we update \( \mathbf{G} \) in \( \Omega_1 = [0, 2] \times [0, 2] \times [1.4, 2.0] \) by using the Huygens–Kirchhoff formula (3.10).

As \( n \) is constant, we can use the exact solution of \( \mathbf{G} \) (see (2.4) in [15]) to validate our numerical solutions. Figures 2 and 3 show our numerical solutions and the exact solutions of \( G_{11} \) and \( G_{31} \) at \( y = 1 \) for \( k_0 = 24\pi \). Detailed comparisons are shown in Figure 4. We see that they match well in the region close to the source and away from it.

Example 2: Gaussian model. In this example, the Green’s function is constructed with the following setup:

- The refractive index function is

\[
 n(x, y, z) = \frac{3}{3 - 1.75e^{-\frac{(x-1)^2+(y-1)^2+(z-1)^2}{6.64}}}.
\]
Fig. 4. Example 1: the real part of $G_{11}$ at (a) line $x = 1$ and $y = 1$ and (b) line $x = \frac{7}{12}$ and $y = 1$; the real part of the $G_{31}$ at (c) line $x = 1$ and $y = 1$ and (d) line $x = \frac{7}{12}$ and $y = 1$. Solid line: the exact solution; circle line: the Huygens sweeping solution. Primary source $r_0 = (1.0, 1.0, 0.2)^T$ and $k_0 = 24\pi$.

Fig. 5. Example 2: real part of $G_{11}$ at $y = 1$ computed by (a) the Huygens sweeping method and (b) the FDTD solution. Primary source $r_0 = (1.0, 1.0, 0.2)^T$ and $k_0 = 10\pi$. Mesh in (a): $97 \times 97 \times 97$; mesh in (b): $401 \times 401 \times 401$.

- The computational domain is $\Omega = [0, 2] \times [0, 2] \times [0, 2]$, and the two-term Hadamard–Babich ingredients are computed on a uniformly spacing grid with size $51 \times 51 \times 51$.
- The primary source point is $r_0 = (1.0, 1.0, 0.2)^T$.
- One secondary source plane is placed at $S_0: z = 1.2$, and we update $G$ in $\Omega_1 = [0, 2] \times [0, 2] \times [1.4, 2.0]$ by using the Huygens–Kirchhoff formula (3.10).

To validate our numerical solutions, we first consider computing $G$ at a low frequency $k_0 = 10\pi$ so that accurate reference solutions can be obtained by the FDTD method. Figures 5 and 6 show our numerical solutions and the FDTD-based solutions of $G_{11}$ and $G_{31}$ at $y = 1$ for $k_0 = 10\pi$. Detailed comparisons are shown in Figure 7. We see that they have good agreement close to the source and away from it. However,
we point out that to attain the same level of resolution of $G$, our asymptotic method requires only $97 \times 97 \times 97$ points, roughly 70 times less than $401 \times 401 \times 401$ points used in the FDTD method.

Moreover, we also use our asymptotic method to compute $G$ at a higher frequency $k_0 = 20\pi$. Figure 8 shows our numerical solutions of $G_{11}$ and $G_{31}$ at $y = 1$ for $k_0 = 20\pi$, where our method requires $193 \times 193 \times 193$ points. Nevertheless, due to the limited CPU and memory resource, the FDTD method is not able to produce accurate results for $G$.

**Example 3: Waveguide model.** In this example, the Green’s function is constructed with the following setup:
Fig. 8. Example 2: the real part of (a) \( G_{11} \) at \( y = 1 \) and (b) \( G_{31} \) at \( y = 1 \), computed by the Huygens sweeping method. Primary source \( r_0 = (1.0, 1.0, 0.2)^T \) and \( k_0 = 20\pi \). Mesh 193 \times 193 \times 193.

Fig. 9. Example 3: the real part of \( G_{11} \) at \( y = 1 \) computed by (a) the Huygens sweeping method and (b) the FDTD solution. Primary source \( r_0 = (1.0, 1.0, 0.2)^T \) and wavenumber \( k_0 = 12\pi \). Mesh in (a): 97 \times 97 \times 77; mesh in (b): 401 \times 401 \times 321.

- The refractive index function is

\[
n(x, y, z) = \frac{1}{1 - 0.5 e^{-6((x-1)^2+(y-1)^2)}}.
\]

- The computational domain is \( \Omega = [0, 2] \times [0, 2] \times [0, 1.6] \), and the two-term Hadamard–Babich ingredients are computed on a uniformly spacing grid with size 51 \times 51 \times 51.
- The primary source point is \( r_0 = (1.0, 1.0, 0.2)^T \).
- Two secondary source planes are placed at \( S_0 : z = 0.6 \) and \( S_1 : z = 1.0 \), and we update \( G \) in \( \Omega_1 = [0, 2] \times [0, 2] \times [0.8, 1.2] \) and \( \Omega_2 = [0, 2] \times [0, 2] \times [1.2, 1.6] \), respectively, by using the Huygens–Kirchhoff formula (3.10).

To validate our numerical solutions, we first consider computing \( G \) at a low frequency so that reference solutions can be obtained by the FDTD method. Figures 9 and 10 show our numerical solutions and the FDTD-based solutions of \( G_{11} \) and \( G_{31} \) at \( y = 1 \) for \( k_0 = 12\pi \). Detailed comparisons are shown in Figure 11. We see that they have good agreement in the region close to the source and away from it. However, we point out that to attain the same level of resolution of \( G \), our asymptotic method requires only 97 \times 97 \times 77 points, roughly 70 times less than 401 \times 401 \times 321 points used in the FDTD method.

Moreover, we also use our asymptotic method to compute \( G \) at a higher frequency \( k_0 = 24\pi \). Figure 12 shows our numerical solutions of \( G_{11} \) and \( G_{31} \) at \( y = 1 \) for \( k_0 = 24\pi \), where we use 193 \times 193 \times 154 points.
5. Conclusion. Based on Hadamard’s method, we extended Babich’s ansatz to the vectorial point-source Maxwell’s equations (1.1). We developed a new Hadamard’s ansatz to form the fundamental solution of the Cauchy problem for the time-domain Maxwell’s wave equations (2.1) and (2.2) in the region close to the source. Governing equations for the unknowns in Hadamard’s ansatz were derived. The initial data for those unknowns were then determined based on a condition of matching Hadamard’s ansatz and the homogeneous-medium Green’s function. Consequently, the Fourier transform of Hadamard’s ansatz in time directly gives the Hadamard–Babich ansatz for the FDPS Maxwell’s equations (2.95) and the Babich-like ansatz (1.3). Finally, incorporating the first two terms of the Hadamard–Babich ansatz into a planar-based Huygens sweeping algorithm, we numerically solved the FDPS Maxwell’s equations.
(1.1) at high frequencies in the region where caustics occur. Numerical experiments demonstrated the accuracy of our method.

REFERENCES


