Joint inversion of surface and borehole magnetic data: A level-set approach

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Running head: Joint magnetic inversion by level set

ABSTRACT

The need to improve the depth resolution of the magnetic susceptibility model recovered from surface magnetic data is a well-known challenge and it becomes increasingly important as the exploration moves to regions under cover and at large depths. Incorporating borehole magnetic data can be an effective means to achieve increased model resolution at depth. The recently developed level-set method for magnetic inversion provides a novel means for constructing the geometrical shape of causative bodies and opens a new avenue for the joint inversion of surface and borehole magnetic data for the purpose of achieving improved depth resolution. We present an extension of the algorithm to the joint inversion and show that the level-set algorithm can resolve the configuration and spatial separation of complex magnetic sources using the information in the magnetic data from sparse boreholes. We further examine the use of borehole intersection information in estimating the crucially important susceptibility values within the context of level-set inversion, and also show that the susceptibility value can also be used as a probing parameter.
to assess the uncertainty in the spatial extent of the causative bodies. Our work demonstrates that the modified level-set inversion leads to an effective means to image and delineate magnetic causative bodies with complex structure by combining the information from surface magnetic data, borehole magnetic data, and sparse drill hole intersection data.
INTRODUCTION

Interpreting magnetic data through 3D inversions plays an important role in many exploration problems. A variety of algorithms have been developed in applied geophysics. Given the limited information content in the magnetic data, much of the research and development on magnetic inversion has focused on the incorporation of prior information to improve the inversion results so that more geological information can be extracted.

One of the challenges in magnetic inversion is how to improve the depth resolution of the recovered source distribution. This challenge arises because magnetic data acquired on or above the earth surface lack inherent depth information in the absence of independent information that helps limit the classes of admissible models. It is common to introduce complementary geophysical data and prior information such as seismic or geological data to constrain the magnetic source depths either implicitly or explicitly.

For instance, Pedersen (1977) and Pilkington and Crossley (1986) utilize the susceptibility contrast between the basin and basement in the basement depth inversion. Bhattacharyya (1980) develops a multibody model represented by a small number of parameters. Wang and Hansen (1990) assume polyhedral causative bodies parametrized by their vertices. Li and Oldenburg (1996) impose a generic depth weighting and a bound constraint in their inversion for 3D susceptibility distributions. Portniaguine and Zhdanov (2002) introduce a minimum support stabilizing functional with the sensitivity weighting. Fullagar et al. (2008) develop a mixed parameterization inversion in which both susceptibility and vertical interface positions may be recovered. In more recent works, Li and Sun (2016) and Sun and Li (2018) constrain the magnetization direction using fuzzy c-means clustering, and Fournier et al. (2016a) and Fournier et al. (2016b) develop an oriented sparse mixed-norm objective function to constrain both the spatial distribution and direction variation of magnetization in the inversion of magnetic data affected by remanent magnetization.

Alternatively, borehole magnetic data can provide supplementary information regarding the depth distribution of magnetic sources. As such, the borehole data directly complement the sur-
face data, and can significantly increase the depth resolution of magnetic interpretation when being combined with surface data. The utility of such complementary information has been amply demonstrated by existing works in the literature. For example, several authors have used simple geometrical methods (e.g., Levanto, 1959; Bosum et al., 1988; Morris et al., 1995) to determine their depth from three-component borehole data. Silva and Hohmann (1981) invert borehole data to recover a single rectangular block. Mueller et al. (1997) jointly interpret borehole magnetic data with those acquired above the surface. Li and Oldenburg (2000) jointly invert borehole and surface magnetic data for 3D susceptibility models by introducing distance- and sensitivity-based weighting functions. Liu et al. (2013) invert borehole magnetic data for total magnetization.

With the development of multiple level-set inversion of magnetic data (Li et al., 2017), there is a new opportunity to combine the complementary information in the surface and borehole magnetic data when inverting for source distributions that are more complicated than isolated simple geometries. The level-set method represents surface or boundary of causative bodies implicitly by the zero-level set of a level-set function. This implicit representation of geometry introduces the possibility to handle topological changes automatically (e.g., Burger and Osher, 2005) during the inversion. With simplifying assumptions adopted by Li et al. (2017), the only site-specific prior information needed is the values of magnetic susceptibility. The re-examination of the formulation also indicates that the algorithm does not depend on explicitly the availability of the data measured above the surface, which is in sharp contrast to the depth weighting (Li and Oldenburg, 1996) formulated specifically for data on or above the surface.

For this reason, we investigate the joint inversion of surface and borehole data using a level-set formulation in this paper. Our work has shown that the joint inversion is able to recover more complex magnetic sources that may be difficult to resolve by the inversion of surface data alone. We also examine the use of information from borehole intersection with magnetic source bodies in estimating susceptibility values. With different trial values of susceptibility, we perform a series of level-set inversions and identify the correct solution by jointly fitting surface magnetic data, borehole magnetic data, and drill hole intersection information.
This paper is organized as follows. First, we review the level-set formulation for the modeling of susceptibility distribution and develop the methodology of level-set joint inversion of surface and borehole magnetic data. We then use two synthetic models with different complexities to illustrate the capability of the joint level-set inversion in resolving vertical magnetic sources when both surface and sparse borehole magnetic data are included. Within that context, we also briefly discuss the exploration of the model space parameterized by the small number of susceptibilities. We then address the practically important issue of how to determine the requisite susceptibility values, or their admissible ranges, for use in the inversion and the uncertainties of recovered models associated with the susceptibility values. Lastly, we present the application of the joint inversion algorithm to a field data set and demonstrate the practical feasibility.

THE LEVEL-SET FORMULATION OF JOINT MAGNETIC INVERSION

The level-set method is a versatile method for surface tracking and shape modeling (Osher and Fedkiw, 2006). Originally introduced by Osher and Sethian (1988), its fundamental idea is to represent a surface or boundary structure \( \Gamma \) via \( \Gamma = \{ x \mid \phi(x) = 0 \} \), where \( \phi(x) \) is the level set function defined everywhere in the model domain. Different from the approaches using explicit parametrization, the level-set method represents surface or boundary implicitly by the zero-level set of the level-set function. This implicit representation of geometry introduces the possibility to handle topological changes such as splitting and merging of connected components in an automatic way (Burger and Osher, 2005). For example, the zero-level set is able to depict any number of surfaces with arbitrary shapes; starting from a single object with connected surface, the evolution of a level-set function can automatically generate multiple disjointed objects. Due to its flexibility in modeling surfaces with complex geometry, the level-set method is widely used as a powerful tool for the solution of shape-optimization and shape-reconstruction problems.

Because of this property, the level-set method has found wide applications in applied geophysical inversion. For instance, Litman et al. (1998) use it in inverse scattering to reconstruct a 2-D binary
obstacle illuminated by time-harmonic electromagnetic line sources; Ito et al. (2001) use it to identify unknown shapes of interfaces in electrical impedance tomography; Hou et al. (2004) propose the level-set method for imaging of location and geometry of extended targets in remote sensing applications; Ben Hadj Miled and Miller (2007) used the method for shape-based reconstruction of conductivity anomaly in electrical resistance tomography; Lewis et al. (2012) and Kadu et al. (2016) used the method for inversion of the geometry of salt bodies in seismic full-waveform inversion. Zheglova et al. (2013) use the level-set method to locate unknown interfaces of seismic medium in the first-arrival traveltime tomography. Li and Leung (2013) propose a level-set adjoint-state method to recover both unknown interfaces and slowness distributions in the problem of transmission traveltime tomography. Li and Qian (2016) propose a level-set-based structural parametrization approach for the joint inversion of gravity and traveltime data. Zheglova et al. (2017) apply a multiple level-set formulation to joint inversion of gravity and traveltime data and investigate applicability of such an inversion method in ore delineation.

In potential fields, the level-set method has shown specific advantages. Isakov et al. (2011) propose an application of the level-set method to the inversion of gravity data. Subsequently Isakov et al. (2013) apply the level-set algorithm to the inverse gravimetry problem for iceberg with snow caps. Lu et al. (2015) then propose an improved fast local level-set method for three-dimensional inverse gravimetry problem by developing two novel algorithms: one designed for speeding up the computation of Frechét derivative in the level-set inversion, and the other for carrying out numerical continuation rapidly to obtain fictitious full measurement data from partial measurement. Lu and Qian (2015) further developed the method for gravity gradient data. Li et al. (2016) apply the level-set method for imaging salt structures using gravity data. Li et al. (2017) develop a multiple level-set method for 3D inversion of surface magnetic data; for more reviews, we also refer readers to that paper.
Level-set representation of magnetic source bodies

To take advantages of the level set method, we use a piecewise-constant model to approximate the susceptibility distribution $\kappa$ in the 3D subsurface. Specifically, we assume that the subsurface geological structure in domain $\Omega$ consists of various magnetic causative bodies with a positive susceptibility contrast against a background that may or may not be magnetic. For brevity, we use the term susceptibility henceforth. Each magnetic body has a constant susceptibility, so that the susceptibility distribution satisfies

$$\kappa(\tilde{r}) = 0, \quad \tilde{r} \in \Omega_0; \quad \kappa(\tilde{r}) = \kappa_i, \quad \tilde{r} \in D_i;$$

$$\Omega_0 \cup (\bigcup_i D_i) = \Omega; \quad D_i \cap D_j = \emptyset, \quad (1)$$

where $\Omega_0$ is the background domain, and $D_i$ is the support of the $i$-th homogeneous causative body with susceptibility contrast $\kappa_i$. The domain $D_i$ is not necessarily connected, which corresponds to the common occurrences of multiple separate causative bodies with the same magnetic susceptibility in a geologic setting.

We propose a level-set approach (van den Doel et al., 2010) to model the unknown susceptibility $\kappa(\tilde{r})$ with piecewise-constant structure separated by interfaces. In general, one can use $n$ level-set functions, with $n$ defined by $2^{n-1} < l \leq 2^n$, in order to describe an object with $l$ distinct constant values. For instance, one may think of $(l - 1)$ homogeneous bodies with different susceptibilities embedded in non-magnetic background, so that the susceptibility distribution consists of $l$ distinct values:

$$\kappa(\tilde{r}) = \begin{cases} \kappa_i, & \tilde{r} \in D_i \ (1 \leq i \leq l - 1) \\ 0, & \tilde{r} \in \Omega \setminus \bigcup_i D_i \end{cases}, \quad (2)$$

Then the level-set formulation is proposed as follows,

$$\kappa(\tilde{r}) = \sum_{t_1, \ldots, t_n = 0,1} C_{t_1, \ldots, t_n} \hat{H}_{t_1} (\phi_1(\tilde{r})) \cdots \hat{H}_{t_n} (\phi_n(\tilde{r})) \quad (3)$$
where $\phi_i(\tilde{r})$ is the level-set function, and

$$\hat{H}_{t_i}(x) = \begin{cases} H(x), & t_i = 0 \\ 1 - H(x), & t_i = 1 \end{cases}$$

with $H(x)$ the Heaviside function:

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}.$$  \hfill (5)

Theoretically, the formulation is able to describe up to $2^n$ distinct susceptibility values with the combination of $n$ indexes $t_1, \cdots, t_n$ taking values either 0 or 1. To reproduce the susceptibility model in formula (2), the coefficients $C_{t_1, \cdots, t_n}$ should take values $\kappa_1, \cdots, \kappa_{l-1}$ and 0; in case that $l < 2^n$ some of the coefficients can be taken the same to arrive at $l$ different values. In practice, however, the lack of prior information would limit the model to only a few distinct values.

It is useful to use the multiple level-set formulation with $n \leq 2$ for the magnetic inverse problem. For example, for the case with two distinct susceptibilities $\kappa_1$ and $\kappa_2$, we use the following level-set formulation,

$$\kappa(\tilde{r}) = \kappa_1 H(\phi_1) (1 - H(\phi_2)) + \kappa_2 (1 - H(\phi_1)) H(\phi_2);$$

in the situation that there is only one type of magnetic source with susceptibility value $\kappa_1$, we use a single level-set formulation:

$$\kappa(\tilde{r}) = \kappa_1 H(\phi_1(\tilde{r})) + 0 \cdot (1 - H(\phi_1(\tilde{r}))) = \kappa_1 H(\phi_1(\tilde{r})).$$

With such representations of magnetic susceptibility distribution in 3D subsurface, we can use standard integral formulations (e.g., Blakely, 1996) to carry out the required forward modeling of the total-field magnetic anomaly data. Given an inducing field of strength $B^0$ and direction $\hat{B}_0$, the anomalous magnetic intensity resulting from a distribution of magnetic susceptibility $\kappa$ is given
by
\[ d(r) = \frac{1}{4\pi} B^0 \int_{\Omega} \kappa(\tilde{r}) \frac{1}{|r - \tilde{r}|^3} \left[ \frac{3 (\hat{B}_0 \cdot (r - \tilde{r}))^2}{|r - \tilde{r}|^2} - 1 \right] d\tilde{r}. \] (8)

where \( \Omega \) is the model domain and the observation location \( r \) can be anywhere above Earth’s surface or in the boreholes. Thus, equation 8 defines the general form of forward modeling for the surface and borehole magnetic data that we are concerned with in this paper. We have assumed that there is no remanent magnetization and that the self-demagnetization effect is negligible. This assumption naturally excludes many problems involving artificial magnetic sources such as well casing or unexploded ordnances, but it is still sufficiently general for many naturally occurring problems in exploration geophysics.

For numerical computation, we discretize the model domain into a 3D grid and convert equation 8 into a matrix-vector form as commonly seen in potential-field inversion (e.g., Li and Oldenburg, 1996). We remark that such a linear system involves a dense coefficient matrix and can incur significant computational cost. To speed up inversions, Li et al. (2017) develop a fast low-rank approximation to the matrix-vector multiplication. We use that approach in this study.

**Level-set joint inversion of borehole and surface data**

We use the method of multiple level-set magnetic inversion presented by Li et al. (2017). The joint inversion problem is solved as an optimization problem in the same manner as the commonly used regularized inversion in applied geophysics:

\[
\begin{align*}
\min. & \quad E_d + \mu E_m \\
\text{s.t.} & \quad E_d = E_d^* 
\end{align*}
\] (9)

where \( E_d \) is the data misfit function, \( E_m \) is the model objective function, and \( \mu \) is the regularization parameter that determines the amount of regularization applied. \( E_d^* \) denotes the optimal value for
the data misfit, and the constraint $E_d = E_d^*$ ensures that the observed magnetic data are well reproduced.

The misfit function $E_d$ is proposed to comprise the information of surface and borehole magnetic data jointly. Denoting the observed surface data and borehole data by \( \{d_{s,k}^*\}_{k=1}^{M_1} \) and \( \{d_{b,k}^*\}_{k=1}^{M_2} \), respectively, and supposing that \( \{d_{s,k}\}_{k=1}^{M_1} \) and \( \{d_{b,k}\}_{k=1}^{M_2} \) are the corresponding predicted data from numerical simulation, we propose the misfit function as follows,

$$E_d = \frac{1}{2(M_1 + M_2)} \left( \sum_{k=1}^{M_1} \alpha_k \left( \frac{d_{s,k} - d_{s,k}^*}{\sigma_{1,k}} \right)^2 + \sum_{k=1}^{M_2} \beta_k \left( \frac{d_{b,k} - d_{b,k}^*}{\sigma_{2,k}} \right)^2 \right), \quad (10)$$

where $\sigma_{1,k}$ and $\sigma_{2,k}$ denote the residual standard deviation of the associated datum; $\alpha_k$ and $\beta_k$ are parameters introduced to control the weight of each datum employed in the inversion algorithm. For simplicity, one can take $\alpha_k = \beta_k = 1$ in most applications, which means that each datum is considered equally in the joint inversion. The factor $1/2(M_1 + M_2)$ is used to normalize the misfit function. If the residuals in the data are Gaussian with zero mean and the corresponding standard deviations are $\sigma_{i,k}$ ($i = 1, 2$), the optimal value for the misfit function is

$$E_d^* = \frac{1}{2(M_1 + M_2)} \left( \sum_{k=1}^{M_1} \alpha_k + \sum_{k=1}^{M_2} \beta_k \right) \quad (11)$$

which is equal to 0.5 if $\alpha_k$ and $\beta_k$ are all taken to be 1. Theoretically, the known value of the expected data misfit allows us to determine the optimal regularization parameter $\mu$ using the discrepancy principle (e.g., Parker, 1994) as per standard approach in regularized inversions.

Equations 10 and 11 and the above discussion are predicated on the assumption of Gaussian noise in the magnetic data. Our experience has been that such an assumption is quite reasonable for surface magnetic data because of the presence of multiple sources of errors from all stages of data acquisition to data processing. This observation is consistent with the expectation from the central limit theorem. It is not clear, however, that the Gaussian assumption is valid to a similar degree for borehole data. For practical applications, this would be an important factor to consider.
When evidences indicate different types of statistical distribution for the data noise, one must adopt a consistent data misfit definition to replace equation 10 and redefine its statistical expectation in equation 11.

We also note that specification of the expected data misfit value (e.g., equation 11) assumes not only a known distribution but also known values of the parameters defining that distribution, such as the standard deviation. When the standard deviation is unknown, we must use independent means to choose the optimal regularization parameter. This is a perennial topic in geophysical inversions and not unique to the level set inversion. Our limited numerical testing indicates that the L-curve criterion (e.g., Hansen, 1992) could be an effective one.

The model objective function \( E_m \) is included to regularize the inversion and produce a model that is geometrically as simple as possible while fitting the magnetic data. We choose to construct a model with a first–order smooth boundary by using the following model objective function:

\[
E_m = \frac{1}{2} \sum_i \int_\Omega |\nabla \phi_i(\tilde{r})|^2 d\tilde{r},
\]  

(12)

where \( \phi_i \) are the level-set functions in the modeling of susceptibility distribution. The objective function \( E_m \) penalizes the variations of the level-set functions. Minimizing this regularization term enables the inversion to smooth the shape of boundary or surface of the sources represented by the zero level sets \( \{ \tilde{r} | \phi_i(\tilde{r}) = 0 \} \). The regularization parameter \( \mu \) is selected to produce the expected data misfit: \( E_d = E^*_d \). In practice, we use the well established approach of carrying out a set of inversions by using different values of \( \mu \) and then performing a line search to find its optimal value (e.g., Hansen, 1992; Aster et al., 2005; Oldenburg and Li, 2005).

We solve the optimization problem in equation 9 by using a gradient-descent approach. The details can be found in Li et al. (2017). For completeness, we also summarize the salient points of the numerical optimization in Appendix-A.

It is important to note that the inversion recovers the level-set functions \( \phi_i \) using the constant
values $\kappa_i$ so that the corresponding 3D susceptibility distribution $\kappa$ produces the optimal data misfit value. In Li et al. (2017), it was assumed that the susceptibility values $\kappa_i$ are given as a priori information. In the situation that the values of $\kappa_i$ are unknown, it becomes essential to estimate them to complete the inversion algorithm. Since the joint inversion of surface and borehole magnetic data provides tools for obtaining more information about the subsurface structure, we are able to investigate the determination of the parameters $\kappa_i$. Although we use a piecewise-constant model to approximate the subsurface susceptibility distribution, the true susceptibility is not necessarily piecewise constant in a precise manner. The parameters $\kappa_i$ model the average susceptibilities of the magnetic sources occupying the domain $D_i$. In reality, it is not always feasible to obtain $\kappa_i$ by direct means such as measuring on rock samples from the magnetic source bodies. We propose instead to use a data-driven strategy to achieve this goal as a part of the inversion, and the primary data are the limited borehole intersections that may be available.

The susceptibility values $\kappa_i$ are determined by the surface magnetic data, borehole magnetic data, and the geometric data from borehole intersections. We will not update $\kappa_i$ and the level-set functions $\phi_i$ simultaneously in the inversion, since algorithmically it is simpler to work with a single class of unknowns such as the level-set functions instead of mixing the susceptibility parameters with the level-set functions. For this reason, we freeze the values of $\kappa_i$, and only evolve the level-set functions $\phi_i$ to reproduce the optimal data misfit value. The geometry of borehole intersections is then employed to evaluate whether the correct susceptibilities are achieved. We perform a sequence of level-set inversions with different trial values of $\kappa_i$, and choose the one that best matches the borehole intersection as the correct solution. We will illustrate the joint inversion algorithm with and without known values of $\kappa_i$ in the numerical examples, and then explore the feasibility of estimating $\kappa_i$ and perform elementary uncertainty quantification.
NUMERICAL EXAMPLES

We now use two numerical examples to illustrate the level-set algorithm for joint inversion of surface and borehole data. The first example has two vertically stacked source bodies and it is designed to examine the increased vertical resolving capability of the level-set inversion in terms of the vertical information from the added borehole magnetic data. The model was derived from conceptualizing a commonly encountered realistic scenario in which magnetic source bodies that are located at different depths produce spatially overlapping anomalies above the surface. The second example has a non-convex causative body and we use it to investigate the advantages provided by combining the flexibility of level-set formulation with such topologies and the geometrical constraints afforded by borehole intersections. This model was derived by conceptualizing an exploration scenario in which a borehole magnetic survey is employed to locate and delineate the intended targets missed by drill holes that are spotted based on the interpretation of surface data.

The magnetic data in both examples are simulated under an inducing field with a strength of $B^0 = 5 \times 10^4$ nT and in the direction with inclination $I = 75^\circ$ and declination $D = 25^\circ$. The surface magnetic data are simulated along the measurement surface $z = -0.1$ km, and there are $21 \times 21$ stations distributed 50-m apart from each other. The borehole data are assumed to be from two boreholes in each case and the stations are spaced 25-m apart. We have excluded the borehole stations near and at the intersections with the magnetic source to simulate the scenario of excluding borehole observations with large errors in those locations (e.g., Li and Oldenburg, 2000). All data have been contaminated with Gaussian noise having a zero mean and a standard deviation of 5 nT. Each datum is equally considered in the inversion process, so that the weighting parameters in equation 10 are set to be $\alpha_k = \beta_k = 1$, and the expected value for the misfit function $E_d$ is 0.5.

We focus on the single level-set formulation in which equation 7 is used to represent the susceptibility distribution. In this section we assume that the susceptibility value $\kappa_1$ is known as a priori information, and we only invert for the unknown level-set function $\phi_1$ by jointly fitting the
surface and borehole magnetic data. We will investigate the determination of susceptibility values in the next section.

**Vertically stacked magnetic sources**

The first example consists of two stacked dipping magnetic sources with susceptibility value $\kappa_1 = 0.04$ SI. Figure 1(a) shows the model, where the vertical lines illustrate locations of drill holes and the stars and circles show borehole measurements. Figure 1(b) displays the cross-section along Easting = 500 m. The surface data are simulated on the measurement plane $z = -0.1$ km, as shown in Figure 1(c). The simulated borehole data are shown in Figure 1(d), in which the data collected from the non-intersecting borehole measurements are denoted by stars and the data from the intersecting borehole measurements are denoted by circles. To perform the level-set inversion, we use an initial guess consisting of a single vertical ellipsoid located at the center of the model domain (Figure 2).

The stacked sources are comparable in dimension and separated vertically only by a distance similar to their thickness, and we further increase the challenge by choosing the shallower source to be larger than the deeper one. Consequently, the magnetic anomalies produced by these two sources overlap in the surface data and appear to be a single compact anomaly. It would be nearly impossible to detect two separated dipping objects by inverting surface magnetic data only, as we shall illustrate next.

As shown in Figure 3(a), the level-set inversion of surface data achieving the expected data misfit of $E_d = 0.5$ produces a single water drop-shaped structure. It indicates the approximate dip and fails to resolve the subjacent dipping source. Figures 3(c) and 3(d) show cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively. They show that this solution recovers the horizontal position and top structure of the shallow source, but does not resolve the depth structure adequately. The normalized differences shown in Figure 3(b) between the predicted and measured surface data are sufficiently random and consistent with the characteristics of the
added noise, and they indicate that the inversion algorithm successfully reproduces the signal in the surface magnetic data. However, the inversion does not resolve two vertically separate source bodies.

We next invert the borehole magnetic data using the level-set inversion algorithm. Figure 4(a) displays the recovered model from the borehole data. Figures 4(c) and 4(d) show two cross-sections through the 3D model. It is clear that the inversion result captures the shape of the dipping source body at depth, and the recovered depth extent agrees well with the true depth extent. The main drawback, however, is that the shape of the shallow source is not adequately reproduced, which consequently affects the overall depth resolution and leads to the appearance of a single connected body instead of two separate ones in the recovered model. Figure 4(b) plots the normalized residual between the predicted and measured borehole data, which corresponds to the expected data misfit of $E_d = 0.5$. The data difference is random as expected, and our algorithm is able to fit the borehole magnetic data. Similarly, however, the sparse borehole data do not have the sufficient information to resolve the two separate bodies.

It is interesting to compare and contrast the respective results from inverting surface and borehole data shown in Figures 3(a) and 4(a). The surface-data inversion is able to image the top of the shallow source and indicate the dipping structure of its top surface, but it is unable to resolve the presence of two vertically separated sources or the depth extent of the source group. The borehole-data inversion is able to image the vertical span of the source group and the dipping features in the source, but it is unable to separate the two sources vertically or resolve the full lateral extent of the shallow source. The information contents in the two data sets are clearly different and complementary. The same observations were made by Li and Oldenburg (2000) when directly inverting for 3D susceptibility models from surface and borehole magnetic data. This leads us logically to the next step of joint inversion to make full use of the complementary information in the two data sets.

Lastly but more interestingly, we jointly invert the surface and borehole magnetic data to recover
a level-set representation of the susceptibility that simultaneously reproduces both data sets. It is also noteworthy that no special modification of the level-set inversion algorithm is needed to apply to the joint inversion of surface and borehole data sets. Figure 5(a) displays the boundary of the recovered model from the joint inversion. The model shows two dipping sources with a clear vertical separation and differing sizes. The same dramatical improvement is observed in the two cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively, shown in Figures 5(b) and 5(c). The inverted model evidently combines the merits of the models from single data set inversions shown in Figures 3 and 4. Both the shallow structure and the depth distribution of the dipping objects are well recovered, and the horizontal and vertical boundaries of two stacked dipping sources are clearly imaged. Figures 5(d) and 5(e) plot the normalized residuals for surface and borehole data, respectively. The data difference is sufficiently randomly distributed, both surface and borehole data are reproduced adequately, and the corresponding data misfit value is again $E_d = 0.5$. We conclude that the level-set inversion has been able to use the complementary information in the two magnetic data sets to produce a superior model.

This example demonstrates the effectiveness of the level-set inversion in delineating causative bodies with complex geometry. Through the joint inversion, we are able to fully explore the capability of the level-set approach and show that surface and borehole data and the complementary information contained therein can be easily and effectively combined to dramatically improve the inversion results. This example also highlights a unique strength of the level set inversion: we use a single ellipsoid as the initial guess assuming no knowledge of multiple sources but the inversion is able to evolve the boundaries to identify two separate bodies.

An interesting question is which region of the subsurface would be improved from the joint inversion of the two data sets in general. As in any joint inversion of multiple data sets, the resultant improvement in the model obtained from joint inversion directly depends on the complementary information provided by different data sets. In this case, the expected improvement in the inverted model will be in the regions to which the surface and borehole data sets have sufficient and complementary sensitivities. Since the borehole data are not linearly dependent upon the surface data,
including the former will almost always provide additional information at some depth range and, therefore, improve the final inverted model.

**Model evaluation using multiple level-set inversions**

The inversion result (Figure 5) assuming a constant susceptibility value of 0.04 yields two separate source bodies located at different depths. In evaluating the results, a few logical questions arise naturally. For example, are there truly two separate bodies? What would be the result if we assumed different susceptibility values for the two recovered bodies?

The first question is simpler to answer. Intuitively, the well known trade-off between the size and the susceptibility would mean that at a sufficiently low susceptibility, the two bodies would expand and connect to form a single object. However, performing a sequence of inversions will readily exclude such a scenario because the large object recovered using too small a susceptibility would have too long an interval of borehole intersection. The inconsistency with the actual borehole intersection data would serve as the constraint to bound the susceptibility from below.

The second question is more interesting and pertinent in practice. The multiple level-set inversion algorithm presented by Li et al. (2017) and used in this study is ideal for answering this question by enabling the exploration of the model in the parameter space of \((\kappa_1, \kappa_2)\), which are the respectively the susceptibility of the shallow and deep body. Although we are seeking to construct a 3D susceptibility model, we only need to examine the variation of the model as a function of the two susceptibility values. Such an exploration of the model space involves the sampling in the 2D parameter space of \((\kappa_1, \kappa_2)\), which is straightforward and numerically tractable.

As an example, we examine a specific aspect of the inverted model shown in Figure 5. The shallower body is noticeably bigger than the deeper one, and both are clearly dipping. Are these features reliable or a mere artifact produced by the assumed susceptibility? We carry out two additional inversions using two different initial models. Both initial models consists of two vertically separate spheres with different susceptibilities. The first assumes an increase of susceptibility by
50% for the shallow body and a reduction of 25% for the deep body, whereas the second has the two values reversed. The inversion results are shown respectively in Figures 6 and 7. All inversions reproduce the data to the same misfit values and we only display the inverted models for clarity.

Several observations can be made without referencing the known true model. First, the level set inversion is able to produce topologically consistent models composed of two separate bodies no matter whether starting from a single ellipsoid or two separate spheres as the initial model. Such consistency lends confidence in the algorithm. Secondly, with three different combinations of susceptibility values (Figures 5, 6, and 7), the shallower body is consistently larger than the deeper one. Therefore, it could be said with a level of confidence that, within the range of susceptibility values explored, the shallower body must be the larger of the two. This is tangible and useful geological information if this were a real-world case. Thirdly, all three models show that the shallow body is dipping and the dip angles estimated from the three models are the same for practical purposes. The same could not be said about the deep body because it has a visible dipping structure when assuming lower susceptibility values but appears to be close to a small vertical object at the high susceptibility value of 0.06.

A doughnut-shaped model

In the second example, we use a steeply dipping doughnut-shaped model to showcase the strength of the level-set inversion for magnetic sources with non-convex geometry. Figure 8(a) shows the model with an opening in the center. Figure 8(b) shows a cross-section of the model along Northing = 500 m with two drillholes. The two boreholes are placed to simulate a hypothetical case where one drill hole intersects the magnetic source near its bottom edge while the second hole misses the source body by going through its opening. The surface data are shown in Figure 8(c) and the borehole data are shown in Figure 8(d), in which the circles indicate data in the vertical borehole and the stars the data in the inclined borehole.

We first invert the surface data starting from an ellipsoidal initial guess as shown in Figure
2. Figure 9(a) shows the inverted model. Figures 9(c) and 9(d) plot the outlines of the magnetic source in two cross-sections of the model, in which the dashed lines indicate the true model and the solid lines show the inversion result. Figure 9(b) shows the normalized data difference between the predicted and measurement surface data with a corresponding data misfit of $E_d = 0.5$. The residual shows that the inversion algorithm adequately reproduces the surface magnetic data. Although the inverted solution is a much smoothed version of the true model and asymmetrically vertical, it images a torus-shaped structure with an opening in the center. Considering the lack of information that directly indicates a non-convex source body and that the inversion started from a solid object as the initial guess, this result is encouraging and showcases the strength of the level-set inversion in working with complex topology. Figure 10 shows a series of intermediate solutions in the inversion process, which illustrates the evolution of the topology in the recovered source body with iterations.

For comparison purposes and to glean some understanding of the possible dependence of the inversion on the initial model, we have performed the inversion of surface magnetic data again using the initial guess of a ring-shaped model shown in Figure 11. The inverted model is consistent with that obtained from the ellipsoidal initial guess. For brevity, we choose not to reproduce the model.

We next examine the inclusion of the borehole magnetic data in the inversion. Since the inversion of the surface data indicates a torus-shaped magnetic source, and one of the two drill holes did not intersect the magnetic source in the center of the source body, it is reasonable to design an initial model that is consistent with the information obtained from the surface inversion and from the borehole intersections. The ring-shaped initial model shown in Figure 11 is a consistent one in that it avoids the location of borehole measurements and also assumes the non-convex topology of the magnetic source based on the inversion result of the surface data.

Using this initial model, we then invert the borehole magnetic data. Figure 12 shows the inversion result, where Figure 12(a) shows 3D structure, Figure 12(c) and 12(d) plot cross-sections of the solution, and Figure 12(b) plots normalized residual of the borehole magnetic data. The inverted structure is significantly different from the true model, although the borehole data are adequately
reproduced as indicated in the plot of the data residual. The inversion of the magnetic data in the borehole has imaged only parts of the source object but indicates also a vertical separation.

We now proceed to jointly inverting the surface and borehole magnetic data. Figure 13(a) shows the recovered solution from the joint inversion, and we observe that the doughnut shape of the source body and its spatial extent are reasonably recovered. Figures 13(b) and 13(c) display the cross-sections along Northing = 500 m and Depth = 400 m, respectively, in which the outlines of the inverted solution (solid curves) match well with those of the true model (dashed lines). Overall, the joint inversion has significantly improved the recovered model. It successfully captures the shape and location of the non-convex doughnut-shaped source body, and the dipping direction of the source body is also imaged. To examine the data fitting, we plot the normalized residual between the predicted and measurement surface data in Figure 13(d) and the normalized residual of borehole data in Figure 13(e). It shows that both data sets are reproduced adequately. Figure 13(e) shows slightly large residuals for a few borehole data points. Such discrepancies are expected for borehole stations located close to the source body.

This example illustrates that the level-set algorithm is capable of resolving complex structures with non-convex configurations that are difficult to achieve in general by explicit parametrization. The joint inversion of surface and borehole data has again produced an inversion result that is superior to those from the inversions of individual data sets.

**DETERMINATION OF SUSCEPTIBILITY VALUES**

A key assumption of the level-set formulation of the magnetic inversion is a prescribed magnetic susceptibility value. A correct value specified a priori removes the trade-off between the strength of the magnetic sources and their sizes or volumes. The removal of the direct trade-off significantly reduces the ambiguity in the inversion and enables us to invert for a reliable representation of the source geometry from the combined surface and borehole magnetic data by incorporating simple generic conditions such as the first-order smoothness. A question of practical importance arises as
to the source of the susceptibility information.

Although in some cases it is feasible to obtain this information from physical property database or from drill logs; alternatively, one may need to estimate the values either independently or during the inversion. As we alluded to earlier, we can include the unknown susceptibility as a part of the level-set inversion algorithm, but we lose certain simplicity and advantage of the method. In this paper, we choose to examine methods to estimate and bound the susceptibility value using additional data such as borehole intersections that are independent of the magnetic data. We refer to this as the data-driven strategy of susceptibility estimation. As expected, such approaches must depend on available information.

We assume that there are limited drill hole intersections or other data available that identify a few points on the magnetic source boundaries. Given the trade-off between the susceptibility and size, it follows that only the inversion result with the correct susceptibility value would be consistent with these known boundary points. Therefore, we perform a sequence of level-set joint inversions using different trial values for the magnetic susceptibility, and choose the one that best matches the configuration of borehole intersections. Thus, the borehole intersections are effectively used as a new data set in the inversion. To illustrate this strategy, we work with the doughnut-shaped model introduced in the preceding section.

As shown in Figures 8(a) and 8(b), the spatial extent of the doughnut-shaped source body is partially constrained between two drill holes. We note that the vertical drill hole to the west is assumed to intersect the magnetic source at one point on the west lower edge, whereas the drill hole to the east does not intersect the source body. The intersection points, or lack thereof, in these drill holes provide useful constraint for the inversion. In fact, the design of a ring-shaped initial guess as shown in Figure 11 is partly guided by this information so that it does not extend beyond the first drill hole and has no intersection with the second.

We perform a sequence of joint inversions using susceptibility trial values $\kappa_1$ starting at 0.01 in increment of 0.01. With $\kappa_1 = 0.01$, the inversion is unable to find an acceptable model that repro-
duces the joint magnetic data set. Figure 14 shows the inversion results for $\kappa_1 = 0.02, 0.03, 0.04$ and 0.05, respectively, in which the left column displays 3D models, and the right column plots the cross-sections at Northing $= 500 \, m$ with dashed straight lines indicating the position of the two drill holes. The inversion result with $\kappa_1 = 0.04$ is shown in the preceding section, and we include it here for completeness. The inversion with too low susceptibility value yields a model that has a broader distribution and exceeds the borehole constraint (e.g., Figures 14(b)). As a result, the data misfit and drill-hole constraints provide a reasonable lower bound for the trial values of the susceptibility. On the other hand, the inverted solution with too high a susceptibility value concentrates the recovered magnetic source into a more compact region that has no intersection with the drill hole. Comparing the length of the intersection between the borehole and inverted model and the actual points of intersection with the true model shown in the right panel of Figure 14, we can conclude that an acceptable value of susceptibility should be between 0.03 and 0.04, which provides a reasonable bracket for the true value $\kappa_1 = 0.04$.

We remark that, although the panels on the right in Figure 14 display the boundaries of the true source bodies, they are displayed for visual presentation only and not used in the above estimation process. The estimation simulates a realistic scenario that we know only the intersection of the drill hole with the west bottom edge of the source body.

All inversions with trial susceptibility values between 0.02 and 0.05 adequately reproduce the surface and borehole magnetic data. In Figure 15, we plot the normalized residual for the surface and borehole data for the inversion with $\kappa_1 = 0.02, 0.03$ and 0.05, respectively. The data residual for the model with $\kappa_1 = 0.04$ is shown in Figure 13. It illustrates that without the knowledge of susceptibility value, there is clearly a trade-off between the size of the source body and the assumed susceptibility value in the magnetic inversion as dictated by physics. The use of borehole intersection as a new data set helps reduce such ambiguity and provides an estimate for the range of possible susceptibility values. In practice, any model within this range may suffice for the purpose of interpretation, or they collectively serve that purpose.
UNCERTAINTY ASSESSMENT OF INVERSION RESULTS

The results in the preceding section clearly indicate that there is a range of realistic and plausible models that constitutes the inverse solution. The key parameter controlling the variability of the models is the susceptibility. Supplying or estimating the susceptibility value can be a challenge in practice and, therefore, constitutes a major source of uncertainty. Viewed from a different perspective, however, the dependence on the susceptibility can be used to our advantage in quantifying the model variability and assessing the uncertainty of the recovered models.

In this section, we present a method to assess the model uncertainty using the inverted model with different trial susceptibility values. The premise is that the inverted solution depends heavily on susceptibility but the changes in the model are nonlinear related to that of susceptibility. Thus, we can turn the susceptibility as a probing parameter to evaluate the model variability in the absence of other constraints such as borehole intersections.

Let us return to the models displayed in the left column of Figure 14, which were obtained by jointly inverting the surface and borehole magnetic data simulated from the doughnut-shaped source body when using trial susceptibility values $\kappa_1 = 0.02, 0.03, 0.04$ and 0.05, respectively. All four models fit the surface and borehole magnetic data adequately, and extending the susceptibility value to below 0.02 or above 0.05 leads to large misfit values and the corresponding models are unacceptable. Thus, this set of models defines both the range of acceptable susceptibility values and, more importantly, the permissible variations in the model. It is also important to note that the four models are highly consistent and overlapping spatially to a large extent. The common overlapping portions are required in all models and can be considered to be required by the data, whereas the differing portions of the models indicate the uncertain region. This approach is similar to the determination of depth of investigation in DC resistivity inversion by using the reference model (Oldenburg and Li, 1999) and the assessment of the uncertainty in geology differentiation through 3D magnetization inversion by using the number of clusters (Li and Sun, 2016).

In our current study, we are concerned with the geometrical shape and spatial extent of the
recovered model. We choose to use the spatial occupancy of the model defined as the spatial locations occupied by a model. This is naturally described by the Heaviside function $H(\phi)$ for each inverted solution: the region occupied by the magnetic source is given by $H(\phi) = 1$ and the outside region not occupied by the source is given by $H(\phi) = 0$. Therefore, we have an occupancy model with binary values for each inverted solution with trial susceptibility value $\kappa_1 \in \{0.02, 0.03, 0.04, 0.05\}$. Intuitively, we can then compute the mean occupancy model and the associated variance from these multiple models. Although these calculations implicitly treat the binary models as if they were continuous variables, the approximation does provide an approximate description of the volume which is most likely occupied by the model. Alternatively, we can also compute a median model from the group of occupancy models given their discrete nature.

Figure 16 shows the detail of the mean occupancy model and the associated standard deviation. To depict the boundary of a magnetic source in the mean occupancy model, we plot the isosurface at the value of 0.9 in Figure 16(a). Figure 16(b) plots the cross-section along Northing = 500 m. Strictly speaking, the mean occupancy model is not an acceptable magnetic inverse solution itself since it is an average of multiple models and there is no associated susceptibility value. However, we do observe that the mean occupancy model’s boundary is consistent with the borehole intersections and fulfills a reasonable representation of the true model. In Figure 16(c), we display the standard deviation of the occupancy model in the cross-section at Northing = 500 m. The standard deviation yields a measure of confidence whether a volume is inside or outside the source body. For example, the two rings in the cross-section with high standard deviation up to 0.6 indicate the regions that could be either the non-magnetic background or part of the magnetic source. In other words, the boundary of the source body lies most likely in these zones. Enclosed in these rings are two regions with low standard deviation and they are definitely part of the source body, whereas the low-standard-deviation region outside the two rings are the background. Figure 17 displays the median occupancy model again plotted as an isosurface at the value of 0.9. Although there are noticeable differences, the median and mean occupancy models are highly consistent in general. Thus, both yield an adequate description of the most likely source region.
A FIELD EXAMPLE

We now apply the joint level-set inversion algorithm to a field data set from Australia. We use this example to illustrate one of the aspects presented in the preceding sections in a real-world case, namely, the improvement in the inverted model that can be gained from including borehole data in a joint inversion. The magnetic source of the anomaly is an ironstone formation hosted in unaltered siltstone and graywacke. Surface magnetic data on a 50-m grid are available for this study. Three-component borehole data were acquired at the site, from which a set of total-field anomaly data was obtained through processing at a 25-m interval in three holes. Figure 18(a) displays the total-field anomaly data above the surface and in three boreholes. Collar locations of the three boreholes are also overlaid on the surface data map. The trajectories of the three boreholes are shown in Figure 18(b) in relation to a sphere to be used as the initial model. Figure 19 displays the total-field anomaly in the three boreholes. The inducing field at the site has inclination $I = -51^\circ$ and declination $D = 5^\circ$. The pattern of the total-field anomaly in the surface data is consistent with that due to induced magnetization. Therefore, the ironstone formation either has no significant remanent magnetization or the remanent magnetization is approximately aligned with the current inducing field direction. Therefore, the data set satisfies the assumptions for the forward modeling based on equation 8 and we can use it to test the joint level-set inversion.

To perform level set inversions, we use a susceptibility value of 0.03 SI, which is estimated based on the work by Li and Oldenburg (2000). Although the susceptibility is not strictly constant within the source body, it is not highly variable. For the purpose of comparing the results produced by the inversions of the surface and joint data sets, respectively, a constant susceptibility value is sufficient. Figure 20(a) displays the difference between observed and predicted surface data. We also show the comparison of observed borehole data and those predicted data in the borehole. Figure 21 displays the result from the inversion of surface data alone. The inverted model reproduces the surface data well as expected, but does not predict the borehole data at all. The model shows an object with a flat lying top portion and a steep extension to the southwest. Overlaid on the cross-section at
N=10,000 m are the composite boundaries of the source body inferred from drill hole intersections. It is clear that the result of surface inversion is inconsistent with the known body geometry.

We next jointly invert the surface and borehole data using the same sphere as the initial model. The inversion achieves the same data misfit value for the surface data and a consistent misfit level for the borehole data. Figure 22 displays the difference between observed and predicted surface data and the comparison of observed and predicted borehole data. Overall, the coherent signals in both surface and borehole data are reproduced consistently. The inverted model from the joint inversion is shown in Figure 23, which is in the same format as Figure 21. The model now shows a clearly dipping object with a simpler structure with a well defined depth extent. Comparing with the known partial boundaries of the body as indicated by the overlaid dashed line, the joint inversion has done a much better job in imaging both the depth and extent of the source body. We emphasize that the inversion only used the surface and borehole magnetic data and the estimated average susceptibility value, but did not include the known boundaries. Thus, the improvement upon the inversion of surface data is completely derived from the information in the borehole magnetic data. This example therefore illustrates the benefit of jointly inverting surface and borehole magnetic data when the condition is suitable and, therefore, shows the efficacy of the joint level-set inversion presented in this paper.

CONCLUSIONS

We have extended a previously-developed level-set inversion algorithm to the joint inversion of surface and borehole magnetic data and studied the advantage of the approach in imaging complex causative bodies with significant vertical variations. Surface magnetic data are predominantly sensitive to the horizontal distribution of sources, whereas data in vertical boreholes are much more sensitive to the vertical source distribution. The level-set inversion provides a natural formulation to combine the complementary information in the two data sets and allows for high-resolution recovery of the geometry of causative bodies in 3D. The ability of the level-set inversion in de-
lineating complex bodies with non-convex geometry makes the method well suited for resolving vertical variations of magnetic sources with the added vertical information from the borehole data. Numerical examples have amply demonstrated these strengths.

Parallel to incorporating borehole magnetic data, we have also examined the possibility of estimating the value of magnetic susceptibility, which is crucially important in the level-set inversion. We have demonstrated that by using the information of sparse borehole intersections, the trade-off between source strength and source volume, which is often stated as a deterring weakness of magnetic data, can be turned into our advantage to estimate the value of magnetic susceptibility or its permissible range of variations. Furthermore, we have also explored the possibility of using the susceptibility value as a probing parameter to appraise the consistency and variance of magnetic models recovered through level-set inversions using a range of susceptibility values. The results indicate that the different models contain consistent and overlapping parts in the spatial distribution and extent. Complementary to the common model, we can also calculate an approximation to the variance of the spatial occupancy of the recovered geometrical model and assess the uncertainties in 3D space. The common model and the variance therefore enable us to form a reliable interpretation from a group of models instead of a single model predicated on the precise knowledge of the susceptibility. Thus, the ability to estimate the susceptibility value and the ability to use it for uncertainty quantification together address an important practical issue of level-set inversion, and make it applicable in cases that were previously thought to be infeasible because of the lack of specific susceptibility value. Therefore, the level-set magnetic inversion can now be used to tackle a wide range of problems in exploration and crustal studies.

APPENDIX A: NUMERICAL SOLUTION OF LEVEL-SET JOINT INVERSION

We solve the optimization problem in equation 9 by using a gradient-descent approach. This appendix describes the details of the numerical solution.
Since the objective function $E_m$ only depends on the level-set functions, there is a straightforward calculation of its gradient:

$$\frac{\partial E_m}{\partial \phi_i} = -\Delta \phi_i,$$

(A-1)

where $\Delta$ denotes the Laplace operator. On the other hand, the gradient computation of the data-misfit function is more involved. As shown in equation 10, the misfit function $E_d$ depends on both the surface magnetic data $\{d_{s,k}\}_{1 \leq k \leq M_1}$ and the borehole magnetic data $\{d_{b,k}\}_{1 \leq k \leq M_2}$. The magnetic data are simulated by equation 8, which is in the form of matrix-vector product after discretization. Let

$$d_s = G_s \kappa \quad \text{and} \quad d_b = G_b \kappa,$$

(A-2)

where $d_s = (d_{s,1}, \cdots, d_{s,M_1})^T$ and $d_b = (d_{b,1}, \cdots, d_{b,M_2})^T$ denote the vectors of surface magnetic data and borehole magnetic data, respectively; $G_s$ and $G_b$ are their corresponding sensitivity matrices whose elements $(G_s)_{k,j}$ or $(G_b)_{k,j}$ quantify the contribution of a unit susceptibility around the $j$th grid point to the $k$th datum. A matrix representation of equation 10 is generated:

$$E_d = \frac{1}{2} \left[ (d_s - d_s^*)^T W_s^T W_s (d_s - d_s^*) + (d_b - d_b^*)^T W_b^T W_b (d_b - d_b^*) \right]$$

(A-3)

with

$$W_s = \frac{1}{\sqrt{M_1 + M_2}} \text{diag} \left\{ \frac{\sqrt{\alpha_k}}{\sigma_{1,k}} \right\}_{1 \leq k \leq M_1} \quad \text{and} \quad W_b = \frac{1}{\sqrt{M_1 + M_2}} \text{diag} \left\{ \frac{\sqrt{\beta_k}}{\sigma_{2,k}} \right\}_{1 \leq k \leq M_2}.$$  

(A-4)

Then the derivative of $E_d$ with respect to $\kappa$ is given by

$$\frac{\partial E_d}{\partial \kappa} = G_s^T W_s^T W_s (d_s - d_s^*) + G_b^T W_b^T W_b (d_b - d_b^*).$$

(A-5)

Since the distance between the surface measurement and the unit susceptibility increases as the depth $\tilde{z}$ increases, the sensitivity matrix $G_s$ for the surface magnetic data decays rapidly with depth. Li et al. (2017) propose a low-rank-matrix decomposition algorithm to take advantage of
this feature. The algorithm partitions the sensitivity matrix $G_s$ according to the depth $\tilde{z}$, and for each submatrix it performs a truncated singular-value decomposition to reduce the complexity of matrix-vector multiplications. This algorithm significantly speeds up the computation of matrix-vector product involving $G_s$, and we will adopt it in this paper. On the other hand, the borehole measurements are located along boreholes which are designed in a more flexible manner according to specific applications. There is not a simple structure of rapid decay in the sensitivity matrix $G_b$, and it can be expensive to perform a partitioned low-rank-matrix decomposition. Moreover, since the borehole measurements are located only in a few sparse holes, the dimension of $G_b$ is much smaller than that of $G_s$. As a result, we perform a direct matrix-vector multiplication for the calculation involving $G_b$. To sum up, we deal with the surface and borehole data separately because their sensitivities have different structures.

Since the susceptibility $\kappa$ is related to the level-set functions $\phi_i$ through equation 7, the derivative of $\kappa$ with respect to $\phi_i$ can be computed point-wise by:

$$
\frac{\partial \kappa}{\partial \phi_i}(\tilde{r}) = \sum_{t_1,\ldots,t_n=0,1} C_{t_1,\ldots,t_n} \hat{H}_{t_1}(\phi_1) \cdots \hat{H}_{t_{i-1}}(\phi_{i-1}) \frac{\partial \hat{H}_{t_i}(\phi_i)}{\partial \phi_i} \hat{H}_{t_{i+1}}(\phi_{i+1}) \cdots \hat{H}_{t_n}(\phi_n), \quad (A-6)
$$

where

$$
\frac{\partial \hat{H}_{t_i}(\phi_i)}{\partial \phi_i} = \begin{cases} 
\delta_d(\phi_i) &, \quad t_i = 0 \\
-\delta_d(\phi_i) &, \quad t_i = 1
\end{cases} \quad (A-7)
$$

with $\delta_d(\cdot)$ denoting the Dirac delta function. For instance, if there are two distinct susceptibilities embedded in non-magnetic background and $\kappa(\tilde{r})$ is formulated by equation (6), the equation (A-6) reduces to

$$
\begin{cases} 
\frac{\partial \kappa}{\partial \phi_1}(\tilde{r}) = \delta_d(\phi_1) \left( \kappa_1 - (\kappa_1 + \kappa_2) \hat{H}(\phi_2) \right) \\
\frac{\partial \kappa}{\partial \phi_2}(\tilde{r}) = \delta_d(\phi_2) \left( \kappa_2 - (\kappa_1 + \kappa_2) \hat{H}(\phi_1) \right)
\end{cases} \quad ; \quad (A-8)
$$

if there is only one type of magnetic source and $\kappa(\tilde{r})$ is formulated by equation 7, then

$$
\frac{\partial \kappa}{\partial \phi_1}(\tilde{r}) = \kappa_1 \delta_d(\phi_1). \quad (A-9)
$$
In numerical computation, we need to introduce a numerical Dirac delta function to approximate $\delta_d(\phi_i)$; for the choice of numerical Dirac delta function, we refer readers to the previous work on level-set methods (e.g., Isakov et al., 2011; Li et al., 2016, 2017). Since $\delta_d(\phi_i)$ vanishes away from the surface of magnetic sources characterized by the zero-level set $\{ \tilde{\mathbf{r}} \mid \phi_i(\tilde{\mathbf{r}}) = 0 \}$, the value of $\frac{\partial \kappa}{\partial \phi_i}(\tilde{\mathbf{r}})$ is non-zero only in a narrow region near the surface structure. Denoting the diagonal matrix composed of the partial derivatives $\frac{\partial \kappa}{\partial \phi_i}$ at each grid point $\tilde{\mathbf{r}}_j$ by

$$\Lambda_i = \text{diag}\left\{ \frac{\partial \kappa}{\partial \phi_i}(\tilde{\mathbf{r}}_j) \right\}_{1 \leq j \leq N} \quad (A-10)$$

and considering equation (A-5), the gradient of $E_d$ is calculated as follows,

$$\frac{\partial E_d}{\partial \phi_i} = \Lambda_i \frac{\partial E_d}{\partial \kappa} = \Lambda_i G_s^T W_s^T W_s (d_s - \mathbf{d}_s^*) + \Lambda_i G_b^T W_b^T W_b (d_b - \mathbf{d}_b^*). \quad (A-11)$$
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LIST OF FIGURES

1 Two stacked dipping objects. (a) true model; (b) cross-section of the model along Easting = 500 m; (c) surface data with Gaussian noise; (d) borehole data with Gaussian noise.

2 An ellipsoidal initial guess for the inversion algorithm.

3 Two stacked dipping objects. Inversion of surface magnetic data. (a) recovered solution; (b) normalized residual in surface data; (c), (d) cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively.

4 Two stacked dipping objects. Inversion of borehole magnetic data. (a) recovered solution; (b) normalized residual in borehole data; (c), (d) cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively.

5 Two stacked dipping objects. Joint inversion of surface and borehole data. (a) recovered solution; (b), (c) cross-sections of the solution along Easting= 500 m and Northing= 500 m, respectively; (d) normalized residual in surface data; (e) normalized residual in borehole data.

6 Two stacked dipping objects. Multiple level-set inversion. $\kappa_1 = 0.06$, $\kappa_2 = 0.03$. (a) initial guess; (b) recovered solution; (c), (d) cross-sections along Easting= 500 m and Northing= 500 m, respectively.

7 Two stacked dipping objects. Multiple level-set inversion. $\kappa_1 = 0.03$, $\kappa_2 = 0.06$. (a) initial guess; (b) recovered solution; (c), (d) cross-sections along Easting= 500 m and Northing= 500 m, respectively.

8 A doughnut-shaped model. True model and magnetic data. (a) A perspective view of the true model; (b) locations of the borehole and borehole measurements in the cross-section along Northing= 500 m; (c) surface magnetic data with Gaussian noise; (d) borehole magnetic data with Gaussian noise.

9 Inversion of surface magnetic data produced by the doughnut model using an ellipsoidal initial guess shown in Figure 2. The inversion has successfully evolved to image the doughnut shape of the true source body. (a) recovered solution; (b) normalized residual in surface data; (c) and (d) the cross-sections of the solution along Northing= 500 m and Depth= 400 m, respectively.
10 A doughnut-shaped model. Intermediate solutions in the inversion of surface magnetic data with an ellipsoidal initial guess. The iteration number is increasing from (a) to (f).

11 A ring-shaped initial guess for the inversion algorithm.

12 Inversion of borehole magnetic data from the doughnut model with a ring-shaped initial guess. (a) recovered solution; (b) normalized residual in borehole data; (c), (d) cross-sections of the solution along Northing= 500 m and Depth= 400 m, respectively.

13 Joint inversion of surface and borehole magnetic data from the doughnut model with a ring-shaped initial guess. (a) recovered solution; (b), (c) cross-sections of the solution along Northing= 500 m and Depth= 400 m, respectively; (d) normalized residual in surface data; (e) normalized residual in borehole data.

14 Joint inversion with different trial values of susceptibility. The left panel shows the recovered solution, and the right panel shows the cross-section along Northing= 500 m. (a),(b) \( \kappa = 0.02 \); (c),(d) \( \kappa = 0.03 \); (e),(f) \( \kappa = 0.04 \); (g),(h) \( \kappa = 0.05 \).

15 Joint inversion with different trial values of susceptibility. The left panel shows normalized residual in surface data, and the right panel shows normalized residual in borehole data. (a),(b) \( \kappa = 0.02 \); (c),(d) \( \kappa = 0.03 \); (e),(f) \( \kappa = 0.05 \).

16 Mean occupancy model evaluated from the inverted solutions with \( \kappa = 0.02, 0.03, 0.04 \) and 0.05. (a) 3D structure of the mean spatial occupancy model shown by the isosurface of 0.9; (b) cross-section of the mean occupancy model along Northing= 500 m; (c) standard deviation of the occupancy models displayed along Northing= 500 m. The high-variance rings define a narrow zone in which the source boundaries must be located for all feasible susceptibility values.

17 Median occupancy model evaluated from the inverted solutions with \( \kappa = 0.02, 0.03, 0.04 \) and 0.05. (a) 3D structure of the median occupancy model shown by the isosurface of 0.9; (b) cross-section along Northing= 500 m.

18 Field data example. (a) surface magnetic data, where the white circles indicate the location of boreholes; (b) initial guess, where the lines illustrate the shape of boreholes.

19 Field data example. Borehole data. (a) data in borehole-2; (b) data in borehole-6; (c)
Data differences in the inversion of surface data. (a) normalized residual of surface data; (b)-(d) comparisons of the observed borehole data with those predicted by the model from surface-data inversion.

Recovered model from the inversion of surface data alone in the field data example. (a) 3D surface of the model viewed from below, (b) cross-sections along Northing= 10000 m, where the dashed lines indicate the known boundary of the source body; (c), (d) plan-sections at Elevation= −150 m and Elevation= −300 m, respectively.

Data differences in the joint inversion of surface and borehole data. (a) normalized residual in the surface data; (b)-(d) comparison between observed and predicted data in the boreholes.

Recovered model from the joint inversion of surface and borehole data. (a) 3D surface of the model viewed from below, (b) cross-sections along Northing= 10000 m, where the dashed lines indicate the known boundary of the source body; (c), (d) plan-sections at Elevation= −150 m and Elevation= −300 m, respectively.
Figure 1: Two stacked dipping objects. (a) true model; (b) cross-section of the model along Easting = 500 m; (c) surface data with Gaussian noise; (d) borehole data with Gaussian noise.
Figure 2: An ellipsoidal initial guess for the inversion algorithm.
Figure 3: Two stacked dipping objects. Inversion of surface magnetic data. (a) recovered solution; (b) normalized residual in surface data; (c), (d) cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively.
Figure 4: Two stacked dipping objects. Inversion of borehole magnetic data. (a) recovered solution; (b) normalized residual in borehole data; (c), (d) cross-sections of the solution along Easting = 500 m and Northing = 500 m, respectively.
Figure 5: Two stacked dipping objects. Joint inversion of surface and borehole data. (a) recovered solution; (b), (c) cross-sections of the solution along Easting $= 500$ m and Northing $= 500$ m, respectively; (d) normalized residual in surface data; (e) normalized residual in borehole data.
Figure 6: Two stacked dipping objects. Multiple level-set inversion. \( \kappa_1 = 0.06, \kappa_2 = 0.03 \). (a) initial guess; (b) recovered solution; (c), (d) cross-sections along Easting = 500 m and Northing = 500 m, respectively.
Figure 7: Two stacked dipping objects. Multiple level-set inversion. $\kappa_1 = 0.03$, $\kappa_2 = 0.06$. (a) initial guess; (b) recovered solution; (c), (d) cross-sections along Easting= 500 m and Northing= 500 m, respectively.
Figure 8: A doughnut-shaped model. True model and magnetic data. (a) A perspective view of the true model; (b) locations of the borehole and borehole measurements in the cross-section along Northing= 500 m; (c) surface magnetic data with Gaussian noise; (d) borehole magnetic data with Gaussian noise.
Figure 9: Inversion of surface magnetic data produced by the doughnut model using an ellipsoidal initial guess shown in Figure 2. The inversion has successfully evolved to image the doughnut shape of the true source body. (a) recovered solution; (b) normalized residual in surface data; (c) and (d) the cross-sections of the solution along Northing= 500 m and Depth= 400 m, respectively.
Figure 10: A doughnut-shaped model. Intermediate solutions in the inversion of surface magnetic data with an ellipsoidal initial guess. The iteration number is increasing from (a) to (f).
Figure 11: A ring-shaped initial guess for the inversion algorithm.
Figure 12: Inversion of borehole magnetic data from the doughnut model with a ring-shaped initial guess. (a) recovered solution; (b) normalized residual in borehole data; (c), (d) cross-sections of the solution along Northing= 500 m and Depth= 400 m, respectively.
Figure 13: Joint inversion of surface and borehole magnetic data from the doughnut model with a ring-shaped initial guess. (a) recovered solution; (b), (c) cross-sections of the solution along Northing = 500 m and Depth = 400 m, respectively; (d) normalized residual in surface data; (e) normalized residual in borehole data.
Figure 14: Joint inversion with different trial values of susceptibility. The left panel shows the recovered solution, and the right panel shows the cross-section along Northing= 500 m. (a),(b) $\kappa = 0.02$; (c),(d) $\kappa = 0.03$; (e),(f) $\kappa = 0.04$; (g),(h) $\kappa = 0.05$. 
Figure 15: Joint inversion with different trial values of susceptibility. The left panel shows normalized residual in surface data, and the right panel shows normalized residual in borehole data. (a),(b) $\kappa = 0.02$; (c),(d) $\kappa = 0.03$; (e),(f) $\kappa = 0.05$. 
Figure 16: Mean occupancy model evaluated from the inverted solutions with $\kappa = 0.02, 0.03, 0.04$ and 0.05. (a) 3D structure of the mean spatial occupancy model shown by the isosurface of 0.9; (b) cross-section of the mean occupancy model along Northing= 500 m; (c) standard deviation of the occupancy models displayed along Northing= 500 m. The high-variance rings define a narrow zone in which the source boundaries must be located for all feasible susceptibility values.
Figure 17: Median occupancy model evaluated from the inverted solutions with $\kappa = 0.02$, 0.03, 0.04 and 0.05. (a) 3D structure of the median occupancy model shown by the isosurface of 0.9; (b) cross-section along Northing$=500$ m.

Figure 18: Field data example. (a) surface magnetic data, where the white circles indicate the location of boreholes; (b) initial guess, where the lines illustrate the shape of boreholes.
Figure 19: Field data example. Borehole data. (a) data in borehole-2; (b) data in borehole-6; (c) data in borehole-7.

Figure 20: Data differences in the inversion of surface data. (a) normalized residual of surface data; (b)-(d) comparisons of the observed borehole data with those predicted by the model from surface-data inversion.
Figure 21: Recovered model from the inversion of surface data alone in the field data example. (a) 3D surface of the model viewed from below, (b) cross-sections along Northing = 10000 m, where the dashed lines indicate the known boundary of the source body; (c), (d) plan-sections at Elevation = −150 m and Elevation = −300 m, respectively.
Figure 22: Data differences in the joint inversion of surface and borehole data. (a) normalized residual in the surface data; (b)-(d) comparison between observed and predicted data in the boreholes.
Figure 23: Recovered model from the joint inversion of surface and borehole data. (a) 3D surface of the model viewed from below, (b) cross-sections along Northing= 10000 m, where the dashed lines indicate the known boundary of the source body; (c), (d) plan-sections at Elevation= −150 m and Elevation= −300 m, respectively.