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An adaptive finite-difference method for traveltimes and amplitudes

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ABSTRACT

The point-source traveltime field has an upwind singularity at the source point. Consequently, all formally high-order, finite-difference eikonal solvers exhibit first-order convergence and relatively large errors. Adaptive upwind finite-difference methods based on high-order Weighted Essentially NonOscillatory (WENO) Runge-Kutta difference schemes for the paraxial eikonal equation overcome this difficulty. The method controls error by automatic grid refinement and coarsening based on a posteriori error estimation. It achieves prescribed accuracy at a far lower cost than does the fixed-grid method. Moreover, the achieved high accuracy of traveltimes yields reliable estimates of auxiliary quantities such as take-off angles and geometric spreading factors.

INTRODUCTION

Many finite-difference methods have been introduced to compute the traveltime for isotropic media directly on regular grids (Reshef and Kosloff, 1986; Vidale, 1988; Podvin and Lecomte, 1991; van Trier and Symes, 1991; Qin et al., 1992; Schneider et al., 1992; Schneider, 1995; El-Mageed et al., 1997; Popovici and Sethian, 1997; Kim and Cook, 1999). The traveltime field is mostly smooth, suggesting that high-order, finite-difference methods should be effective. The use of upwind differencing in all of the cited methods confines the errors to singularities which develop away from the source point. However, the source point itself is an upwind singularity. The truncation error of a pth-order method is dominated by the product of $(p+1)$ derivatives of the traveltime field and the $(p+1)$ power of the step(s). The $(p+1)$ derivatives of the traveltime field, however, behave like the $(-p+1)$ power of the distance to the source, since in the constant-velocity case traveltime is equal to distance divided by velocity. Therefore, near the source—when the distance is on the order of the step—the truncation error is quadratic in the step, i.e., first order. This inaccuracy spreads throughout the computation and renders all higher-order methods first-order convergent. Moreover, the resultant inaccuracy in traveltime prevents reliable computation of auxiliary quantities such as take-off angles and amplitudes.

This inaccuracy affects all point-source traveltime computations using gridded eikonal solvers. In the few published convergence tests, implementers have resorted to imposing a grid-independent region of constant velocity near the source in which the traveltimes are initialized analytically. This is the approach taken by Sethian (1999) in demonstrating second-order convergence for a version of his fast marching method. The approach has two obvious drawbacks: (1) the velocity may not be homogeneous near the source and (2) the size of the region of analytic computation must be set by the user and bears no obvious relation to the grid parameters. In principle, highly accurate ray-tracing methods could be used to alleviate the first difficulty, but the second remains: it introduces an arbitrary parameter into the use of eikonal solvers. Kim and Cook (1999) take a different approach, similar to the one we advocate. They refine the grid several times near the source so the reduced grid spacing compensates for the increased truncation error. However, their grid refinement strategy appears to be adhoc, and it once again involves an arbitrary parameter—namely, the number of grid refinements near the source—without a clear selection criterion.

In this paper, we show how to use adaptive-gridding concepts commonplace in the numerical solution of ordinary differential equations (Gear, 1971) to resolve the difficulty caused by this inaccuracy. Adaptive gridding has already been used in numerical solutions of PDEs (Berger and Oliger, 1984; Berger and LeVeque, 1998). Generally, the grid refinement must be localized in several dimensions, leading to complex data structures. Fortunately, the nature of the traveltime field permits a relatively straightforward adaptive gridding strategy (Belfi and Symes, 1998). The present work improves that of Belfi and

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Symes through the use of the more accurate Weighted Essentially NonOscillating (WENO) difference scheme and extends it to solutions of advection equations for various geometric acoustics quantities. The efficiencies achieved by the adaptive gridding are considerable—usually more than an order of magnitude reduction in computation time for problems of typical exploration size, compared to fixed-grid methods giving the same level of accuracy. We also obtain dramatic improvements in the accuracy of computed geometric acoustics quantities, such as take-off angles and geometric amplitudes.

The essential principle of adaptive gridding is simple. It is based on a hierarchy of difference schemes of various orders. Presumably a higher-order step is more accurate than a lower-order step, so the higher-order step can serve as a substitute for the exact solution when evaluating the local error in the lower-order step. Therefore, one can combine the step computations of two different orders to obtain a so-called a posteriori estimate of the truncation error for the lower-order step. Since the lower- and higher-order truncation errors stand in a known asymptotic relation, this permits an estimate of the higher-order truncation error as well. The asymptotic form of the truncation error then permits prediction of a step that will result in a lower-order truncation error less than a user-specified tolerance. As long as the steps are selected to maintain this local error, standard theory predicts that the higher-order global error, i.e., the actual error in the solution computed using the higher-order scheme, will be proportional to the user-specified tolerance. This straightforward idea is embedded in most modern software packages for solutions of ordinary differential equations (ODEs) (Gear, 1971). Its use for partial differential equations (PDEs) is a little more complicated because it is usually necessary to adjust the grid of the nonevolution variables along with the evolution step. As first established by Belfi and Symes (1998), the solution of the (paraxial) eikonal equation changes in a sufficiently predictable way to make grid adjustment practical.

The paper begins with a description of paraxial eikonal equations for traveltimes. Then we formulate the advection equation for take-off angles and present the amplitude formulas for a 2-D line source and point source. We briefly describe numerical schemes needed in the adaptive gridding approach, presenting the details in Appendix A. With these ingredients in place, we introduce the adaptive gridding principle for the eikonal equation with a point source and present a simple implementation. Numerical experiments demonstrate that the new approach gives us not only accurate traveltime fields but accurate amplitude fields as well. We conclude with some discussion on adaptive gridding in the 3-D case.

PARAXIAL EIKONAL EQUATIONS

The traveltime field in an isotropic solid satisfies an eikonal equation. Denote by (x_s, z_s) the coordinates of a source point and by (x, z) the coordinates of a general point in the subsurface. The first-arrival traveltime field $\tau(x, z; x_s, z_s)$ is the viscosity solution of the eikonal equation (Lions, 1982),

$$\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = s^2(x, z), \tag{1}$$

with the initial condition

$$\lim_{(x,z)\to(x_s,z_s)} \left(\frac{\tau(x, z; x_s, z_s)}{\sqrt{(x - x_s)^2 + (z - z_s)^2}} - \frac{1}{v(x, z)}\right) = 0,$$

where v is velocity and $s = 1/v$ is slowness.

In some seismic applications, the traveltime field is needed only in regions where

$$\frac{\partial \tau}{\partial z} \geq s \cos\theta_{\max} > 0,$$

i.e., along downgoing, first-arriving rays making an angle $\leq \theta_{\max} < \pi/2$ with the vertical. To enforce this condition, we modify the eikonal equation as an evolution equation in depth (Gray and May, 1994):

$$\frac{\partial \tau}{\partial z} = H\left(\frac{\partial \tau}{\partial x}\right) = \sqrt{\mathrm{smmax}\left(s^2 - \left(\frac{\partial \tau}{\partial x}\right)^2, s^2 \cos^2\theta_{\max}\right)}, \tag{2}$$

where smmax is a smoothed max function defined by

$$\mathrm{smmax}(x, a) = \begin{cases} \frac{1}{2}a & \text{if } x < 0 \\ \frac{1}{2}a + 2\frac{x^4}{a^3}\left(1 - \frac{4}{5}\frac{x}{a}\right) & \text{if } 0 \leq x < \frac{a}{2} \\ x + 2\frac{(x-a)^4}{a^3}\left(1 + \frac{4}{5}\frac{x-a}{a}\right) & \text{if } \frac{a}{2} \leq x < a \\ x & \text{if } x \geq a \end{cases},$$

where $a > 0$ (Qian et al., 1999).

Equation (2) defines a stable nonlinear evolution in z, suitable for explicit finite-difference discretization. The smoothed max function makes the numerical Hamiltonian smooth enough to carry out standard truncation error analysis for schemes of up to third-order accuracy. The solution τ is identical to the solution of the eikonal equation provided that the ray makes an angle $\leq \theta_{\max} < \pi/2$ with the vertical. If the ray makes an angle $> \theta_{\max}$ with the vertical, the corresponding wavefront is replaced by an artificial plane wave.

ADVECTION EQUATIONS FOR TAKE-OFF ANGLES

Based on the traveltime computed by solving the eikonal equation, we can approximate the amplitude field by solving a transport equation. The amplitude satisfies the following transport equation (Cerveny et al., 1977):

$$\nabla\tau \cdot \nabla A + \frac{1}{2}A\nabla^2\tau = 0. \tag{3}$$

Equation (3) is a first-order advection equation for the amplitude A. The Laplacian of the traveltime field is involved in this advection equation, which implies that we need a third-order accurate traveltime field to get a first-order accurate amplitude field (Symes, 1995; El-Mageed et al., 1997).

For convenience in the following presentation, we introduce the ray coordinates. The ray coordinates are defined by $(\tau, \phi) = (\tau(x, z; x_s, z_s), \phi(x, z; x_s, z_s))$, where τ and ϕ are the traveltime and take-off angle of a ray from source point (x_s, z_s) to a general point (x, z) in the subsurface. In 2-D isotropic media with line sources, the amplitude also satisfies the formula (Cerveny et al., 1977; Friedlander, 1958)

$$A = \frac{v}{2\pi\sqrt{2}}\sqrt{|\nabla\tau \times \nabla\phi|}, \tag{4}$$

where $\nabla\phi$ and $\nabla\tau$ are the gradients of the take-off angle and the traveltime, respectively.

Since the take-off angle ϕ is constant along any ray,

$$\nabla\tau \cdot \nabla\phi = \frac{\partial\tau}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\tau}{\partial z}\frac{\partial\phi}{\partial z} = 0. \tag{5}$$

That is, the wavefront normal $\nabla\tau$ is tangential to the ray; the gradient $\nabla\phi$ is tangential to the wavefront. Equation (5) is slightly easier to solve numerically than equation (3) because no second-order traveltime derivatives are explicitly involved in equation (5). Having solved equation (5) for ϕ, one produces the amplitude A through equation (4).

Since the typical seismic source is a point source, we need to compensate for the out-of-plane radiation in the 2-D line-source amplitude formula. The 2-D amplitude with a point source (2.5-D amplitude) can be computed by

$$A = \frac{v}{4\pi}\sqrt{\tau_{yy}|\nabla\tau \times \nabla\phi|}, \tag{6}$$

where the out-of-plane curvature τ_{yy} satisfies another advection equation (Symes et al., 1994),

$$\frac{\partial\tau}{\partial x}\frac{\partial\tau_{yy}}{\partial x} + \frac{\partial\tau}{\partial z}\frac{\partial\tau_{yy}}{\partial z} + \tau_{yy}^2 = 0. \tag{7}$$

Supposing that the amplitude is required to be first-order accurate, the two gradients $\nabla\tau$ and $\nabla\phi$ involved in the amplitude formulas should have at least first-order accuracy. However, because after discretization of equation (5) $\nabla\phi$ depends on second-order derivatives of traveltime τ, it implies that to get a first-order accurate $\nabla\phi$ the traveltime τ itself should have at least third-order accuracy. The final conclusion is that a third-order traveltime solver is required to get first-order accurate amplitudes, as noted before.

Zhang (1993) uses equation (6) in polar coordinates to compute the geometric spreading factor, but his computation of the take-off angle is based on the first-order traveltime field. Consequently, the gradient of take-off angle computed by his scheme is inaccurate. Vidale and Houston (1990) encounter a similar difficulty.

FINITE-DIFFERENCE SCHEMES

The literature suggests a large number of competing finite-difference and related schemes to solve the eikonal equation. We use the Essentially NonOscillatory (ENO) schemes (Osher and Sethian, 1988; Osher and Shu, 1991) and the related Weighed ENO (WENO) schemes (Liu et al., 1994; Shu, 1998; Jiang and Peng, 2000) for the following reasons: (1) stable schemes of arbitrarily high-order accuracy exist, permitting accurate solutions on coarse grids (a factor critical to the mesh refinement or coarsening) and (2) versions exist in any dimension so that we can straightforwardly extend our methodology to the 3-D case (El-Mageed et al., 1997; Kim and Cook, 1999).

All of these schemes take the form of recursive depth stepping rules:

$$\tau \leftarrow \tau + \delta_n^n\tau, \tag{8}$$

$$z \leftarrow z + \Delta z. \tag{9}$$

Here, δ_n^n is a nonlinear update operator expressing the WENO–Runge-Kutta rule of order n, defining a difference scheme of formal nth-order accuracy and depending on Δz, Δx, and the slowness field s. Since we want to emphasize the strategy of the adaptive gridding approach, we put the detailed form of δ_n^n $(n = 2, 3)$ in Appendix A.

Similarly, we solve the advection equation for the take-off angle ϕ and the out-of-plane curvature τ_{yy} by using WENO schemes.

ADAPTIVE GRIDDING IMPLEMENTATION

To initialize our algorithm, the user supplies a local error tolerance ϵ; σ_1 and σ_2 are two user-defined positive functions of ϵ to control coarsening and refinement. For example, we can take $\sigma_1 = 0.1\epsilon$ and $\sigma_2 = \epsilon$. We use the second- and third-order eikonal solvers [equations (A-1) and (A-2)] and estimate the truncation error of the second-order scheme as $e_2 = \max|\delta_2^2\tau - \delta_3^3\tau|$ over the current depth. As long as $\sigma_1(\epsilon) \le e_2 \le \sigma_2(\epsilon)$ at every point of the current depth level, we simply proceed to the next step. It is well known (Gear, 1971) that for ordinary differential equations, an efficient adaptive stepping requires rather loose control of the local error. Hence, the factor of 10 difference between σ_1 and σ_2 is reasonable and works pretty well in practice. When $e_2 < \sigma_1(\epsilon)$, we increase the step by a factor of two, i.e., $\Delta z \leftarrow 2\Delta z$, and we recompute the τ update and e_2. Similarly, when $e_2 > \sigma_2(\epsilon)$, we decrease the step by a factor of two. As soon as the local error is once again within the tolerance interval, we continue depth stepping. A very important point is that we retain the third-order (a more accurate one) computation of τ at the end of each depth step as the actual update, discarding the second-order computation, which is used only in step control.

The usual step adjustment in ODE solvers would change Δz by a factor computed from the asymptotic form of the truncation error (Stoer and Bulirsch, 1992, 499). This is impractical for a PDE application because it would require an arbitrary adjustment of the spatial grid (i.e., the x-grid in the difference scheme) and therefore expensive interpolation. Scaling Δz by a factor of two, however, implies that the stability may be maintained by scaling Δx by the same factor. For coarsening, this means throwing out every other grid point, i.e., no interpolation at all, which dramatically reduces the floating-point operations required. Since the typical behavior of the traveltime field is to become smoother as one moves away from the source, the truncation errors generally tend to decrease. Therefore, most of the grid adjustments are coarsenings and very little or no interpolation is required. Since the slowness field comes to us in gridded form, an interpolation is always required to supply estimates of slowness at the points appearing in the WENO–Runge-Kutta formula. We use a local quadratic interpolation in x and z because the third-order accuracy is compatible with that of the difference scheme. For traveltimes, we use a similar quadratic interpolation.

Since the traveltime field is not smooth at the source point, the truncation error analysis on which the adaptive step selection criterion is based is not valid there. Therefore, it is necessary to produce a smooth initial traveltime field. We do this by estimating the largest $z_{\text{init}} > 0$ at which the constant velocity traveltime is in error by $< \sigma_2(\epsilon)$. Details of the z_{init} calculation are given in Appendix B. Having initialized τ at z_{init}, the algorithm invokes adaptive gridding. Since z_{init} is quite small, τ changes rapidly, resulting in a large number of grid refinements at the outset. However, no interpolation is performed because τ is given analytically on $z = z_{\text{init}}$. This initially very fine grid is rapidly coarsened as depth stepping proceeds.

In our current implementation, we maintain a data structure for the computational grid that is independent of the output grid; the desired quantities are calculated on the computational grid and interpolated back to the output grid. As a safeguard against pathological program behaviors, we specify a maximum number of permitted grid refinements, MAXREF.

A simplified algorithm framework is as follows.

Input ϵ, x_s, z_s, $\theta_{\max}$, Δz, MAXREF.
Initialize Δx, τ, $z = z_{\text{init}}$, Ref $= 0$.
Do while $z <$ target depth:

- compute $e_2 = \max|\delta_2^2 \tau - \delta_3^3 \tau|$ over the current depth level z;
- if $e_2 \le \sigma_1(\epsilon)$ and REF > 0,
  * $\Delta z \leftarrow 2\Delta z$,
  * $\Delta x \leftarrow 2\Delta x$,
  * REF $\leftarrow$ REF $- 1$,
  * upsample τ (throw out every other point)
- else if $e_2 \ge \sigma_2(\epsilon)$ and REF $\le$ MAXREF,
  * $\Delta z \leftarrow \Delta z/2$,
  * $\Delta x \leftarrow \Delta x/2$,
  * REF $\leftarrow$ REF $+ 1$,
  * downsample τ (interpolate)
- else
  * $z \leftarrow z + \Delta z$,
  * $\tau \leftarrow \tau + \delta_3^3 \tau$
- end if

end do.

This description leaves out the output step of the algorithm: a full implementation monitors the depth level of the next set of output points and quadratically interpolates the traveltime field onto them as soon as z passes this depth, using the current and last two depth levels of τ. Local quadratic interpolation preserves the third-order accuracy of the computed τ.

To avoid unnecessary computations, we update τ only within the triangle

$$\{(x, z) : |x - x_s| \le |z - z_s| \tan\theta_{\max}\}.$$

All rays with take-off angles $<\theta_{\max}$ must lie inside this triangle, and it is only along such rays that the paraxial eikonal equation produces correct first-arrival times. Output points outside the triangle are assigned a very large number so that constructed raypaths will never reach those places. Because traveltimes at output points inside the triangle but not lying on rays with take-off angles $<\theta_{\max}$ also receive erroneous time values, they must be washed out of any subsequent computations. For high-frequency asymptotics computations, this masking is most easily accomplished by zeroing the geometric amplitude at such points.

NUMERICAL EXPERIMENTS

To illustrate how the adaptive gridding approach works, we test our method on a constant-velocity model $v = 1$ km/s, with 2-D geometry $\{(x, z) : -0.5\,\text{km} \le x \le 0.5\,\text{km}, 0 \le z \le 1.0\,\text{km}\}$, where the behaviors of traveltime fields and amplitude fields are well understood.

In the constant-velocity case, all the desired quantities have obvious analytic solutions to compare against the computed solutions. We compare the results obtained by fixed and adaptive grid algorithms. Both algorithms use a third-order WENO scheme (Appendix A) to compute τ; the adaptive grid scheme uses a second-order ENO scheme to estimate local truncation error. The ouput grid is 51×51, with $\Delta x = \Delta z = 0.02$ km. For adaptive grid algorithms, MAXREF is set to be 5, with the coarsest grid 17×17, $\sigma_1 = 0.1\epsilon$, and $\sigma_2 = \epsilon$.

Frist, we compare the computation cost of the two methods. Tables 1 and 2 show the traveltime error and computation cost by the fixed and adaptive grid methods, respectively, where Flops denote the number of floating point operations. The error is the maximum absolute error at the bottom row of the gridpoints ($z = 1$ km). The computed portion of this depth level ($-$ 0.5 km $\le x \le$ 0.5 km) lies entirely within the computation aperture ($\theta_{\max} = 78^\circ$) and so consists of accurately computed τ values. We can see that to reach the same level of accuracy, the adaptive-gridding approach requires an order of magnitude lower computational cost than does the fixed gridding approach.

Second, we illustrate the difference of accuracy of the two algorithms. For the fixed gridding algorithm, the computational grid is 200×200, with $dx = 0.005$ km. For the adaptive gridding algorithm, the local error tolerance ϵ is 0.00001. The traveltime contours (not shown) produced by the two approaches have no obvious difference because the fixed gridding algorithm still has first-order accuracy. Figure 1 shows contours of τ_x computed by two approaches. We can see that τ_x by the fixed grid is oscillating but τ_x by the adaptive grid traveltime solver is convergent. Because the fixed gridding approach gives us only first-order accurate traveltime field, the resultant traveltime derivatives have only zero-order accuracy and exhibit oscillations that do not decrease in magnitude as the grid is refined (Figure 1a). However, the adaptive gridding approach yields far more accurate traveltime fields; thus, the traveltime derivatives are still accurate (Figure 1b). Similar phenomena are observed for τ_z.

Now we discuss the take-off angle and its derivatives. Because the coefficients in the advection equation for take-off angles depend on the traveltime gradient, the accuracy of ϕ is decided by the traveltime solver we use. Since the first-order traveltime field from the fixed gridding approach results in inaccurate $\nabla\tau$, the resultant take-off angle is not accurate enough to be differentiated. However, the take-off angle based on the traveltime field from the adaptive gridding approach is accurate enough to be differentiated. Figure 2 shows ϕ_x by the two

Table 1. Fixed-grid eikonal solver: a constant velocity model.

dx	Absolute error (τ, dx)(s)	Flops
0.01	0.001232	261 590
0.00125	0.000219	16 632 765

Table 2. Adaptive-grid eikonal solver: a constant-velocity model.

ϵ	Absolute error (τ, dx)(s)	Flops
0.000025	0.001041	39 815
0.00000169	0.000160	928 770

approaches. Because the take-off angle based on the traveltime field from the fixed gridding approach is inaccurate, the resultant derivatives ϕ_x are divergent (Figure 2a). However, the adaptive gridding approach produces accurate traveltime gradients, which leads to the convergent ϕ_x (Figure 2b). Similar observations hold for ϕ_z.

To further illustrate the differences of the accuracy between two approaches, Figure 3 shows the distribution of relative errors along the depth direction for ϕ_x. The error along the depth direction is defined as

$$e(z) = \frac{\max_{-0.5\le x\le 0.5} | f^{\text{comp}}(x,z) - f^{\text{ana}}(x,z)|}{\max_{-0.5\le x\le 0.5} | f^{\text{ana}}(x,z)|}, \quad (10)$$

where f^{comp} is the computed solution and f^{ana} is the analytic solution. For instance, substituting f with ϕ_x in Equation (10), we get the error distribution for ϕ_x along the depth direction. From Figure 3, we can conclude that the adaptive gridding approach produces much more accurate $\nabla\phi$ than does the fixed gridding approach. The resultant amplitudes with a line source based on $\nabla\tau$ and $\nabla\phi$ by the two approaches are shown in Figure 4; one is divergent by the fixed gridding approach, the other is accurate by the adaptive gridding approach. Note the episodic nature of the convergence for the adaptive gridding algorithms. Because we have allowed the local error estimate to vary by an order of magnitude before adjusting the grid and then permitted only step changes by factors of 2, the error exhibits sticky, discontinuous behaviour.

Finally, Figure 5 shows the computational results for the out-of-plane curvature τ_{yy} and the amplitude field with the point source by the adaptive gridding approach. The computed τ_{yy} is accurate, and the resultant amplitude is convergent.

We have embedded the adaptive grid traveltime and amplitude solver in 2-D Kirchhoff prestack migration and inversion code (Symes et al., 1994). Figure 6 shows the impulse response of the inversion for a WENO third-order eikonal solver, where the Beylkin determinant required by the inversion is computed by using the information from traveltimes and take-off angles. We will report the complete test result of the new adaptive traveltime and amplitide solver embedded in migration and inversion in a future paper.

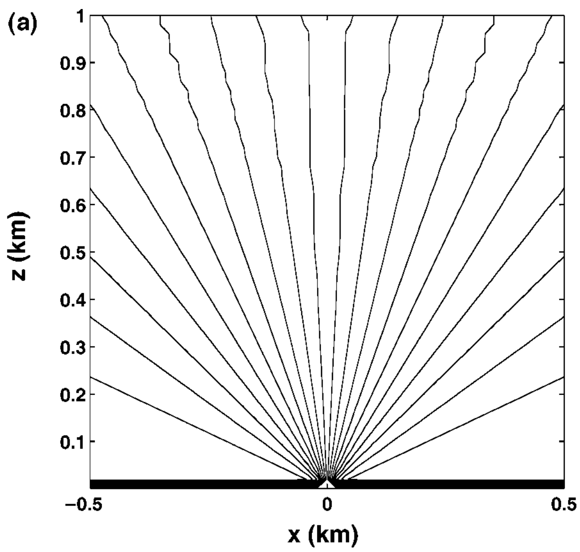

FIG. 1. The value τ_x for a constant velocity model. (a) τ_x by fixed grid is oscillating. (b) τ_x by adaptive grid is convergent.

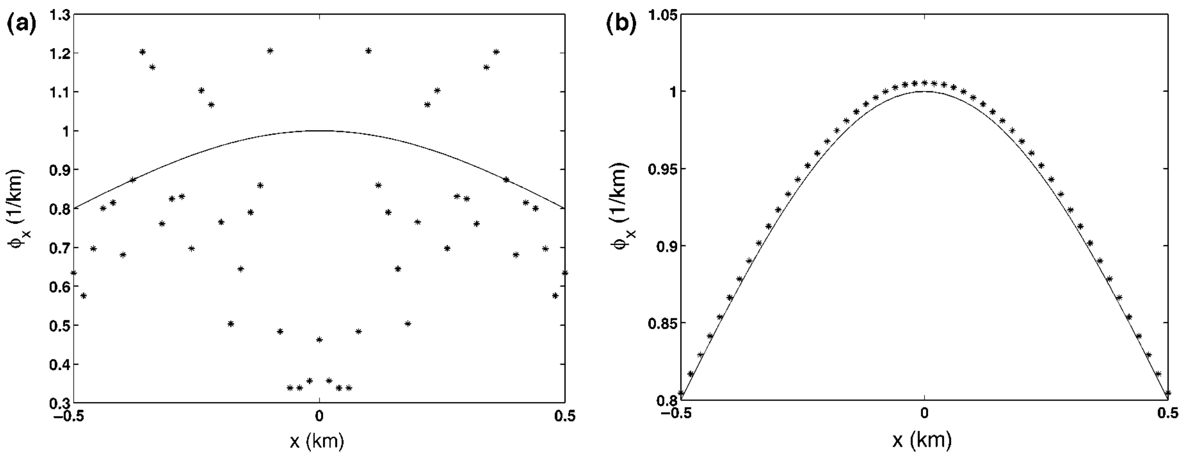

FIG. 2. The value ϕ_x at $z = 1$ for a constant-velocity model. (a) Fixed grid; solid line (−)—true solution; star (*)—computed solution. (b) Adaptive grid; solid line (−)—true solution; star (*)—computed solution.

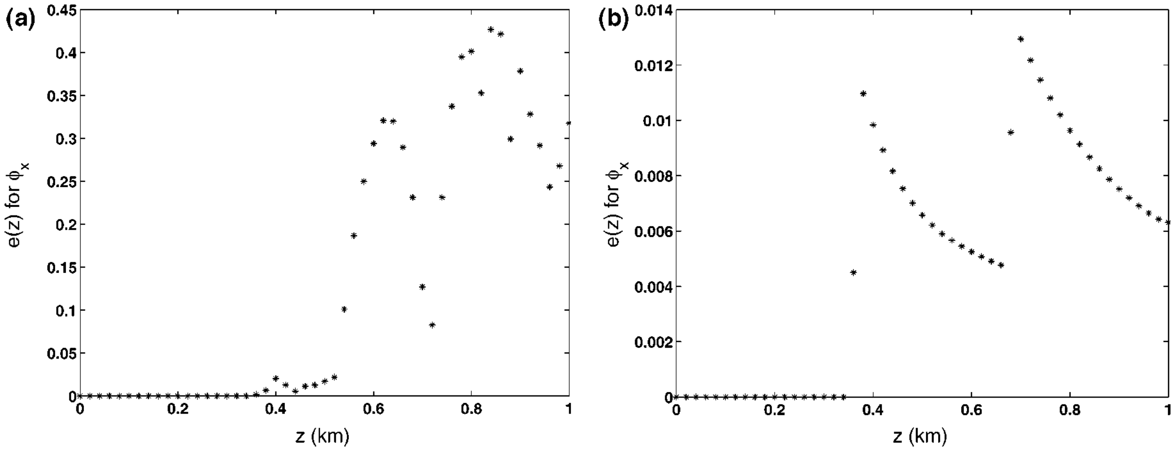

FIG. 3. Relative errors in ϕ_x. (a) Fixed grid—maximum relative error is almost 45%. (b) Adaptive grid—maximum relative error is <1.5%.

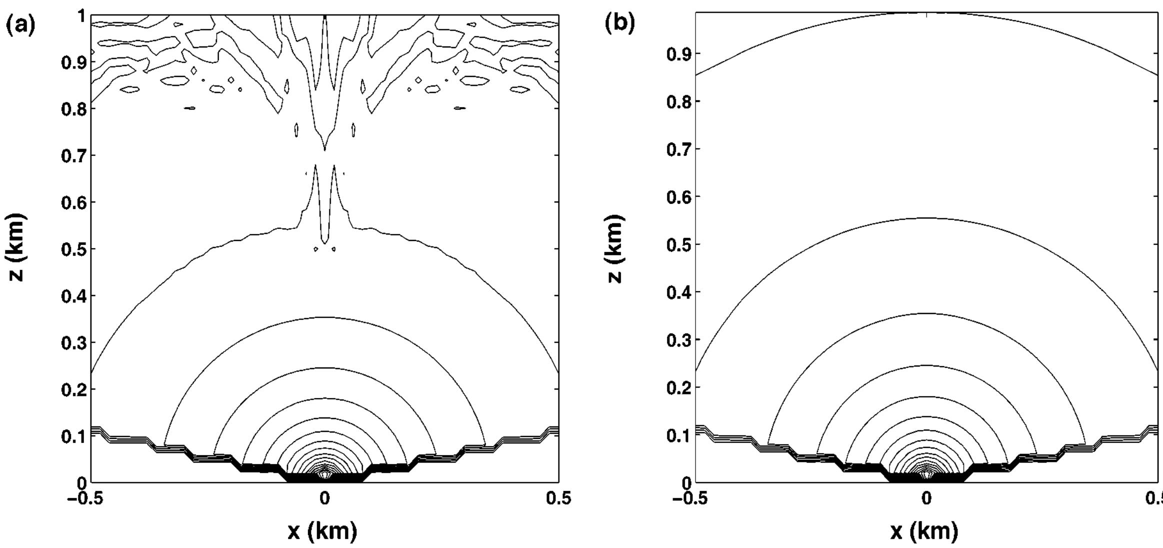

FIG. 4. 2-D amplitude with a line source for a constant-velocity model. (a) The amplitude by fixed grid is divergent. (b) The amplitude by adaptive grid is convergent.

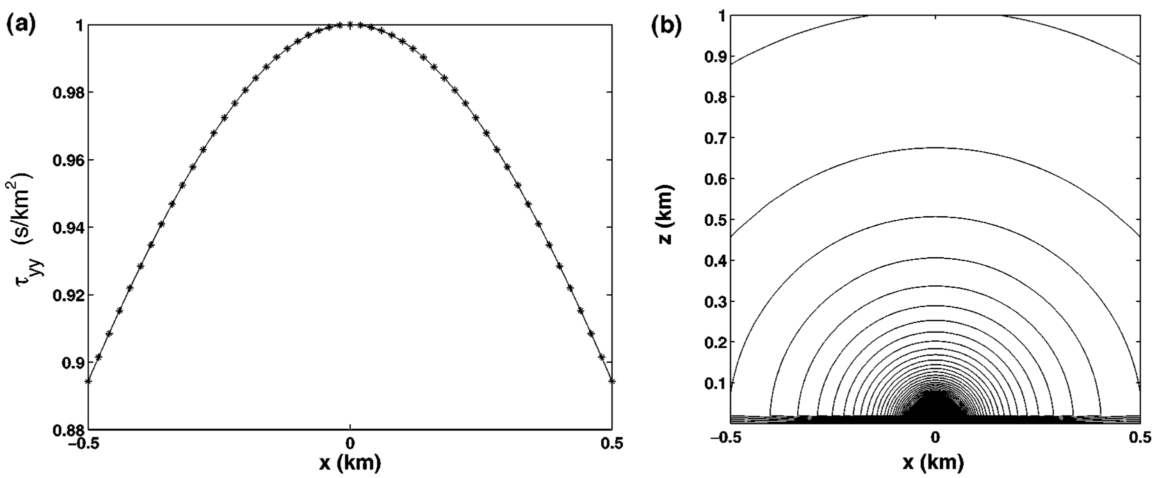

FIG. 5. (a) τ_{yy} at $z = 1$ for a constant-velocity model by adaptive grid; solid line (−)—true solution; star (*)—computed solution. (b) 2-D amplitude with a point source for a constant-velocity model by adaptive grid.

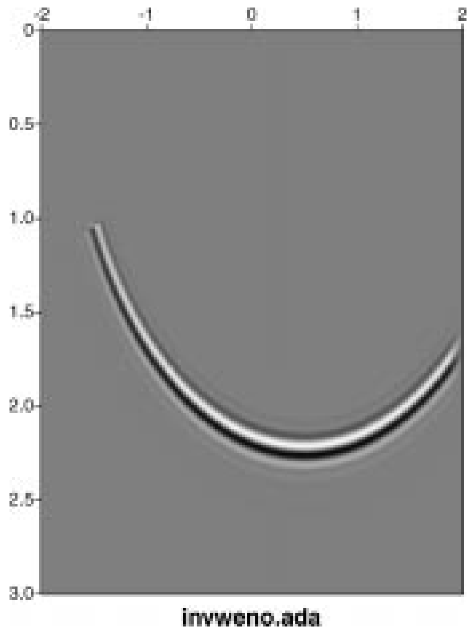

FIG. 6. The impulse response by inversion with adaptive gridding WENO traveltime—amplitude solver. The Beylkin determinant needed in the inversion is computed by using the information from traveltimes and take-off angles, and the response is smooth as expected.

CONCLUSIONS

In this paper we stated a paraxial eikonal equation with depth as evolution direction and an advection equation for take-off angles. Then we presented high-order WENO difference schemes to solve the eikonal equation for the traveltime and the advection equation for the take-off angle. To deal with the singularity of a point source, we proposed a new adaptive grid eikonal solver and detailed the implementation. Numerical experiments showed that the new method yields an efficiency gain of more than an order of magnitude in computational time. Adaptive gridding does not altogether eliminate the arbitrary parameter feature, for which we criticized other approaches in the introductory section of this paper; however, our arbitrary parameter is the local-error tolerance ϵ. In principle, ϵ is proportional to the (global) error in the computed solution, but the relation is complex (as the numerical example shows) and not a byproduct of the algorithm. Nonetheless, we maintain that the simplicity and homogeneity of the algorithm and the direct if not apparent relation between ϵ and the global solution error make the adaptive grid scheme easier to use than its alternatives. Also, the considerable success of the variable-step selection methods for ODEs, which have the same indirect error control feature, supports this contention.

The extension to 3-D isotropic media is straightforward. Because all of the difference schemes presented here can easily be extended to the 3-D case, there is no difficulty in implementing a 3-D version of the adaptive traveltime and amplitude solver. Moreover, we expect the efficiency gain in computational cost will be even more dramatic in the 3-D case. Furthermore, we already extended the adaptive gridding algorithm to computing traveltimes and amplitudes in anisotropic media (Qian, 2000). A fully adaptive eikonal solver based upon a posteriori error estimates for general numeric methods for Hamilton-Jacobi equations (Albert et al., 2002) will be the subject of a subsequent paper.

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REFERENCES

Albert, S., Cockburn, B., French, D., and Peterson, T., 2002, A posteriori error estimates for general numerical methods for Hamilton-Jacobi equations, part I—The steady state case: Math. Comp., **71**, 49–76.

Belfi, C., and Symes, W. W., 1998, An adaptive ENO algorithm for the eikonal equation: Annual Report, The Rice Inversion Project (http://www.trip.caam.rice.edu/).

Berger, M., and LeVeque, R. J., 1998, Adaptive mesh refinement using wave-propagation algorithms for hyperbolic systems: SIAM J. Numer. Anal., **35**, 2298–2316.

Berger, M., and Oliger, J., 1984, Adaptive mesh refinement for hyperbolic partial differential equations: J. Comput. Phys., **53**, 484–512.

Cerveny, V., Molotkov, I. A., and Psencik, I., 1977, Ray method in seismology: Univ. Karlova Press.

El-Mageed, M. A., Kim, S., and Symes, W. W., 1997, 3-D Kirchhoff migration using finite difference traveltimes and amplitudes: Annual Report, The Rice Inversion Project (http://www.trip.caam.rice.edu/).

Friedlander, F., 1958, Sound pulses: Cambridge Univ. Press.

Gear, C. W., 1971, Numerical initial value problems in ordinary differential equations: Prentice-Hall, Inc.

Gray, S., and May, W., 1994, Kirchhoff migration using eikonal equation traveltimes: Geophysics, **59**, 810–817.

Jiang, G. S., and Peng, D., 2000, Weighted ENO schemes for Hamilton-Jacobi equations: SIAM J. Sci. Comput., **21**, 2126–2143.

Kim, S., and Cook, R., 1999, 3-D traveltime computation using second-order ENO scheme: Geophysics, **64**, 1867–1876.

Lions, P. L., 1982, Generalized solutions of Hamilton-Jacobi equations: Pitman Advanced Publishing Program.

Liu, X. D., Osher, S. J., and Chan, T., 1994, Weighted essentially nonoscillatory schemes: J. Comput. Phys., **115**, 200–212.

Osher, S. J., and Sethian, J. A., 1988, Fronts propagating with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations: J. Comput. Phys., **79**, 12–49.

Osher, S. J., and Shu, C. W., 1991, High-order essentially nonoscillatory schemes for Hamilton-Jacobi equations: SIAM J. Num. Anal., **28**, 907–922.

Podvin, P., and Lecomte, I., 1991, Finite difference computation of traveltimes in very contrasted velocity models: A massively parallel approach and its associated tools: Geophys. J. Internat., **105**, 271–284.

Popovici, A. M., and Sethian, J. A., 1997, Three-dimensional traveltime computation using the fast marching method: 67th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1778–1781.

Qian, J., 2000, Geometrical optics for Quasi-P waves: Theories and numerical methods: Ph.D. thesis, Rice University, Houston, TX77251-1892.

Qian, J., Belfi, C. D., and Symes, W. W., 1999, Adaptive finite difference method for traveltime and amplitude: 69th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1763–1766.

Qin, F., Luo, Y., Olsen, K. B., Cai, W., and Schuster, G. T., 1992, Finite difference solution of the eikonal equation along expanding wavefronts: Geophysics, **57**, 478–487.

Reshef, M., and Kosloff, D., 1986, Migration of common shot gathers: Geophysics, **51**, 324–331.
Schneider, W. A. J., 1995, Robust and efficient upwind finite-difference traveltime calculations in three dimensions: Geophysics, **60**, 1108–1117.
Schneider, W. A. J., Ranzinger, K., Balch, A., and Kruse, C., 1992, A dynamic programming approach to first arrival traveltime computation in media with arbitrarily distributed velocities: Geophysics, **57**, 39–50.
Sethian, J. A., 1999, Level set methods, 2 ed.: Cambridge Univ. Press.
Shu, C. W., 1998, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, *in* Cockburn, B., Johnson, C., Shu, C., and Tadmor, E., Eds., Advanced numerical approximation of nonlinear hyperbolic equations: Lecture notes in mathematics, **1697**, Springer Pub., Co. Inc. 325–432.
Stoer, J., and Bulirsch, R., 1992, Introduction to numerical analysis, 2 ed.: Springer-Verlag, New York Inc.
Symes, W. W., 1995, Mathematics of reflection seismology: Annual Report, The Rice Inversion Project (http://www.trip.caam.rice.edu/).
Symes, W. W., Versteeg, R., Sei, A., and Tran, Q. H., 1994, Kirchhoff simulation, migration and inversion using finite difference traveltimes and amplitudes: Annual Report, The Rice Inversion Project (http://www.trip.caam.rice.edu/).
van Trier, J., and Symes, W. W., 1991, Upwind finite-difference calculation of traveltimes: Geophysics, **56**, 812–821.
Vidale, J., 1988, Finite-difference calculation of travel times: Bull. Seis. Soc. Am., **78**, 2062–2076.
Vidale, J. E., and Houston, H., 1990, Rapid calculation of seismic amplitudes: Geophysics, **55**, 1504–1507.
Zhang, L., 1993, Imaging by the wavefront propagation method: Ph.D. thesis, Stanford University.

APPENDIX A

WENO SCHEMES FOR EIKONAL EQUATIONS

Our adaptive scheme is based on the second- and third-order WENO difference schemes introduced by Jiang and Peng (2000). These in turn are extensions of second- and third-order ENO difference schemes, which we present first.

For a function f of the space variable (x, z) in the computational domain, we write

$$f_i^k = f(x_i, z_k),$$
$$(x_i, z_k) = (x_{\min} + (i-1)\Delta x, z_{\min} + (k-1)\Delta z).$$

Let

$$\tau_i^k = \tau(x_i, z_k; x_s, z_s)$$

and define the forward D^+ and backward D^- finite-difference operators

$$D_x^{\pm}\tau_i^k = \frac{\pm\left[\tau_{i\pm1}^k - \tau_i^k\right]}{\Delta x}.$$

The second- and third-order ENO refinements of $D_x^{\pm}\tau$ are

$$\begin{aligned} D_x^{\pm,2}\tau_i &= D_x^{\pm}\tau_i \mp \frac{1}{2}\Delta x\, m\left(D_x^{\pm}D_x^{\pm}\tau_i,\, D_x^{-}D_x^{+}\tau_i\right),\\ D_x^{\pm,3}\tau_i &= D_x^{\pm,2}\tau_i - \frac{1}{6}(\Delta x)^2\\ &\quad\times m\left(D_x^{\pm}D_x^{\pm}D_x^{\pm}\tau_i,\, D_x^{+}D_x^{+}D_x^{-}\tau_i,\, D_x^{+}D_x^{-}D_x^{-}\tau_i\right), \end{aligned}$$

where

$$\begin{aligned} m(x,y) &= \min(\max(x,0), \max(y,0))\\ &\quad + \max(\min(x,0), \min(y,0)). \end{aligned}$$

Similar refinements exist for any order.

The upwind ENO approximations for $\partial\tau/\partial x$ are

$$\hat{D}_x^n\tau = \operatorname{modmax}\left(\max\left(D_x^{-,n}\tau, 0\right), \min\left(D_x^{+,n}\tau, 0\right)\right),$$

where the modmax function returns the larger value in modulus.

The second and third-order ENO–Runge-Kutta steps are

$$\begin{aligned} \delta_2^1\tau &= \Delta z H\left(\hat{D}_x^2\tau\right),\\ \delta_2^2\tau &= \frac{1}{2}\left(\delta_2^1\tau + \Delta z H\left(\hat{D}_x^2\left(\tau + \delta_2^1\tau\right)\right)\right), \end{aligned} \tag{A-1}$$

and

$$\begin{aligned} \delta_3^1\tau &= \Delta z H\left(\hat{D}_x^3\tau\right),\\ \delta_3^2\tau &= \frac{1}{4}\left(\delta_3^1\tau + \Delta z H\left(\hat{D}_x^3\left(\tau + \delta_3^1\tau\right)\right)\right),\\ \delta_3^3\tau &= \frac{1}{3}\left(2\delta_3^2\tau + 2\Delta z H\left(\hat{D}_x^3\left(\tau + \delta_3^2\tau\right)\right)\right). \end{aligned} \tag{A-2}$$

The depth step Δz must satisfy the stability condition

$$\Delta z \le \Delta z_{\text{cfl}} = \frac{\Delta x}{\tan(\theta_{\max})}.$$

We have typically chosen $\Delta z = 0.9\Delta z_{\text{cfl}}$.

The nth-order scheme is then

$$\tau^{k+1} = \tau^k + \delta_n^n\tau^k \tag{A-3}$$

for $k = 0, 1, 2, \ldots$.

We have observed that the gradient of the take-off angle based on the third-order ENO traveltime is too noisy to lead to an accurate amplitude field. To alleviate this phenomenon, instead of ENO third-order refinements, we use WENO third-order refinement (Jiang and Peng, 2000) to compute $D_x^{\pm}\tau$ in the third-order Runge-Kutta step, which yields an accurate amplitude field.

The WENO third-order schemes for $D_x^{\pm}\tau_i$ are

$$\begin{aligned} D_x^{\pm\mathrm{W},3}\tau_i &= \frac{1}{12}\left(-D_x^{+}\tau_{i-2} + 7D_x^{+}\tau_{i-1} + 7D_x^{+}\tau_i - D_x^{+}\tau_{i+1}\right)\\ &\quad \pm \Delta x\,\Phi^{\mathrm{W}}\left(D_x^{-}D_x^{+}\tau_{i\pm2}, D_x^{-}D_x^{+}\tau_{i\pm1}, D_x^{-}D_x^{+}\tau_i,\right.\\ &\qquad \left.D_x^{-}D_x^{+}\tau_{i\mp1}\right), \end{aligned}$$

where

$$\begin{aligned} \Phi^{\mathrm{W}}(a,b,c,d) &= \frac{1}{3}w_0(a - 2b + c)\\ &\quad + \frac{1}{6}\left(w_2 - \frac{1}{2}\right)(b - 2c + d) \end{aligned}$$

with weights defined as

$$\begin{aligned} w_0 &= \frac{\alpha_0}{\alpha_0+\alpha_1+\alpha_2}, \quad w_2 = \frac{\alpha_2}{\alpha_0+\alpha_1+\alpha_2},\\ \alpha_0 &= \frac{1}{(\delta+\beta_0)^2}, \quad \alpha_1 = \frac{1}{(\delta+\beta_1)^2}, \quad \alpha_2 = \frac{1}{(\delta+\beta_2)^2},\\ \beta_0 &= 13(a-b)^2 + 3(a-3b)^2,\\ \beta_1 &= 13(b-c)^2 + 3(b+c)^2,\\ \beta_2 &= 13(c-d)^2 + 3(3c-d)^2. \end{aligned}$$

In the denominators above, we added a small positive number δ to avoid dividing by zero. In the computation, δ is chosen to be 10^{-6}. In practice, the solution is not senstive to the choice of δ.

Next, we have to compute the take-off angle ϕ and out-of-plane curvature τ_{yy}. To match with the evolution form of the eikonal equation in depth, we formulate the advection equation for take-off angles as an evolution equation in depth as well, i.e.,

$$\frac{\partial\phi}{\partial z} = -\left(\frac{\partial\tau}{\partial z}\right)^{-1}\frac{\partial\tau}{\partial x}\frac{\partial\phi}{\partial x}. \qquad \text{(A-4)}$$

To take full advantage of the accuracy of traveltimes produced by the WENO Runge-Kutta third-order scheme for the eikonal equation and simplify the implementation, we embed the third-order scheme for equation (A-4) into the third-order scheme for the eikonal equation. Because the coefficient of the discretized advection equation has only second-order accuracy, which is computed from the eikonal equation by the third-order scheme, we use a second-order upwind WENO scheme to approximate the derivatives $\partial\phi/\partial x$. The advection equation for τ_{yy} is treated similarly. See Qian (2000) for details.

APPENDIX B

ESTIMATE THE INITIAL STEP

To initialize the traveltime for finite-difference schemes, we assume that the velocity near the source is constant and equal to the velocity at the source. Now we desire to analyze the traveltime error resulting from this assumption and compute an a priori estimate of the initial step.

Assuming that the source is at the origin, we consider the 2-D ray-tracing equation. By the method of characteristics for the eikonal equation, we have

$$\dot{x} = v^2 p, \qquad \text{(B-1)}$$

$$\dot{z} = v^2 q, \qquad \text{(B-2)}$$

$$\dot{p} = -\frac{1}{v}\frac{\partial v}{\partial x}, \qquad \text{(B-3)}$$

$$\dot{q} = -\frac{1}{v}\frac{\partial v}{\partial z}, \qquad \text{(B-4)}$$

where the dot $(\cdot)$ denotes the differentiation with respect to time t along the ray, $p = \partial\tau/\partial x$, and $q = \partial\tau/\partial z$.

Denoting the group angle as θ, we have

$$\dot{x} = v\sin\theta, \qquad \text{(B-5)}$$

$$\dot{z} = v\cos\theta. \qquad \text{(B-6)}$$

Furthermore, equations (B-1) and (B-2) yield

$$p = \frac{\sin\theta}{v(x,z)}, \qquad \text{(B-7)}$$

$$q = \frac{\cos\theta}{v(x,z)}. \qquad \text{(B-8)}$$

Differentiating equation (B-7) with respect to time t and simplifying the resultant equation, we have

$$\dot{\theta} = -\cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial z}. \qquad \text{(B-9)}$$

Now we introduce polar coordinates, i.e.,

$$x = r\sin\psi, \qquad \text{(B-10)}$$

$$z = r\cos\psi. \qquad \text{(B-11)}$$

Differentiating equations (B-10) and (B-11) with respect to time t and solving for $\dot{r}$ and $\dot{\psi}$, we have

$$\dot{r} = v\cos(\theta - \psi), \qquad \text{(B-12)}$$

$$\dot{\psi} = \frac{v}{r}\sin(\theta - \psi). \qquad \text{(B-13)}$$

Next we want to estimate $(\theta - \psi)$. First of all, we have $|\theta - \psi| < \pi$, since for the downward-wave propagation both θ and ψ lie in the interval $(-\pi/2, \pi/2)$. Defining

$$a(t) = \dot{\theta}, \qquad \text{(B-14)}$$

$$b(t) = \frac{vt}{r}\frac{\sin(\theta - \psi)}{(\theta - \psi)}, \qquad \text{(B-15)}$$

by equations (B-9) and (B-13) we have an ordinary differential equation for $(\theta - \psi)$,

$$\dot{\theta} - \dot{\psi} = a(t) - \frac{b(t)}{t}(\theta - \psi). \qquad \text{(B-16)}$$

Its solution is

$$\theta - \psi = \int_0^t d\tau\, a(\tau)\exp\left(-\int_\tau^t d\sigma\frac{b(\sigma)}{\sigma}\right). \qquad \text{(B-17)}$$

Because $b(t) \geq 0$ and the function a is bounded by $a_{\max}$, which is equal to the supremum of the length of gradient of the velocity, i.e., $|a| \leq a_{\max}$, equation (B-17) yields an estimate for $\theta - \psi$:

$$|\theta - \psi| \leq a_{\max} t. \qquad \text{(B-18)}$$

Now we are ready to get an approximate relative error estimate for the traveltime. Denote t_0 as the approximation to the exact traveltime t when we are using the constant velocity v_0 at the source as the approximation to the exact velocity v. Since

$$\dot{t}_0 = \frac{\dot{r}}{v_0} = \frac{v}{v_0}\cos(\theta - \psi), \qquad \text{(B-19)}$$

we have

$$\dot{t}_0 - \dot{t} = \left(\frac{v}{v_0} - 1\right)\cos(\theta - \psi) + \cos(\theta - \psi) - 1. \qquad \text{(B-20)}$$

Furthermore,

$$|\dot{t}_0 - \dot{t}| \leq \left|\frac{v}{v_0} - 1\right| + |\cos(\theta - \psi) - 1|. \qquad \text{(B-21)}$$

If $|\dot{t}_0 - \dot{t}| \leq \varepsilon$, then $|t_0 - t| \leq \varepsilon t$. So let's specify that

$$\left|\frac{v}{v_0} - 1\right| \leq \frac{\varepsilon}{2} \qquad \text{(B-22)}$$

and

$$|\cos(\theta - \psi) - 1| \leq \frac{\varepsilon}{2}. \qquad \text{(B-23)}$$

Expanding v at the origin (the source) by Taylor theorem with remainder, we have

$$v(x,z) = v_0 + \frac{\partial v}{\partial x}(\zeta_1,\eta_1)x + \frac{\partial v}{\partial z}(\zeta_2,\eta_2)z, \qquad \text{(B-24)}$$

where (ζ_1,η_1) and (ζ_2,η_2) lie in

$$D = \{(\zeta,\eta) : \min(x,0) \le \zeta \le \max(x,0), 0 \le \eta \le z\}. \qquad \text{(B-25)}$$

As a consequence,

$$|v(x,z) - v_0| \le \sqrt{2}r \sup\{|\nabla v(\zeta,\eta)| : |\zeta| \le |x|, 0 \le \eta \le z\} \qquad \text{(B-26)}$$

by Cauchy inequality.

Because we are only bounding the error inside the aperture,

$$|x| \le z\tan\theta_{\max}, \quad r \le \frac{z}{\cos\theta_{\max}}. \qquad \text{(B-27)}$$

It follows that

$$\left|\frac{v - v_0}{v_0}\right| \le \frac{\sqrt{2}r}{v_0}\sup\{|\nabla v(\zeta,\eta)| : |\zeta| \le z\tan\theta_{\max}, 0 \le \eta \le z\} \le \frac{\sqrt{2}zB}{v_0\cos\theta_{\max}}, \qquad \text{(B-28)}$$

where $z_{\max}$ is the maximum depth and

$$B = \sup\{|\nabla v(\zeta,\eta)| : |\zeta| \le z_{\max}\tan\theta_{\max}, 0 \le \eta \le z_{\max}\}. \qquad \text{(B-29)}$$

For equation (B-22) to hold, by equation (B-28) we should choose z such that

$$z \le z_1 = \frac{v_0\varepsilon\cos\theta_{\max}}{2\sqrt{2}B}. \qquad \text{(B-30)}$$

Finally we choose z so that equation (B-23) holds, and we need a lemma to do so.

Lemma 1.—*Along a ray segment* $\{(x(\tau), z(\tau)) : 0 \le \tau \le t\}$, *the following inequality holds:*

$$t \le \frac{r}{v_{\min}}, \qquad \text{(B-31)}$$

where $r = \sqrt{x^2(t) + z^2(t)}$; $v_{\min}$ *is the minimum velocity along the ray segment.*

Proof.—Denote the true raypath as s and its length $|s|$, and the straight raypath as l and its length $|l|$, which is equal to r. In addition, use l to approximate the true raypath s. Then by Fermat's principle, we have

$$t = \int_s d\sigma\frac{1}{v} \le \int_l d\sigma\frac{1}{v} \le \int_l d\sigma\frac{1}{v_{\min}} = \frac{r}{v_{\min}}. \qquad \text{(B-32)}$$

Using equation (B-18) and lemma 1, we have

$$|\cos(\theta - \psi) - 1| = \left| -2\sin^2\frac{(\theta-\psi)}{2}\right| \le \frac{r^2B^2}{v_{\min}^2}, \qquad \text{(B-33)}$$

where we use the relation $a_{\max} \le B$ inside the aperture. Hence, to make equation (B-23) hold implies that

$$z \le z_2 = \sqrt{\frac{\varepsilon}{2}}\frac{v_{\min}\cos\theta_{\max}}{B}. \qquad \text{(B-34)}$$

So for error tolerance ε, z_{init} should be chosen such that

$$z_{\text{init}} = \min(z_1, z_2). \qquad \text{(B-35)}$$

Although both z_1 and z_2 depend on B (the bound of gradient of velocity model), there are at least two ways to estimate B. One way is setting B to be a big number that is larger than the actual value. The other way is computing the gradient of velocity model from the given discretized model. Both ways produce a reasonable initial step.