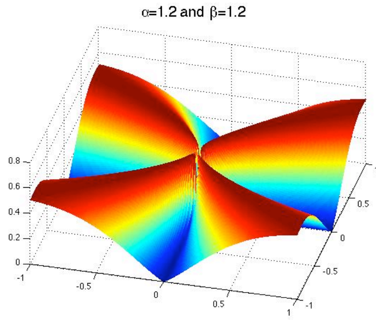
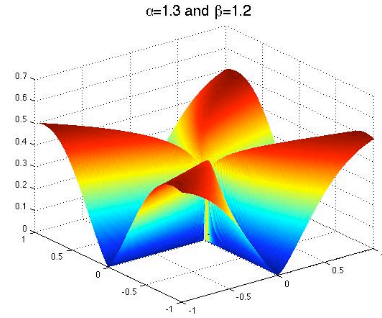


M421 HW 5

Due Fri. Nov 14



From Wade

Section	Page Number	Problems
11.1	392	4, 5
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Non-book Exercises

1) For which $\alpha > 0$ is the function

$$f(x,y) = \begin{cases} \dfrac{x^2|y|^\alpha}{x^2+|y|^3} & (x,y) \neq 0 \\ 0 & (x,y) = 0, \end{cases}$$

differentiable at zero?

Honor's Problems

2) Define the space $\mathcal{C}^1[a,b] = \{f : [a,b] \mapsto \mathbf{R} \Big| \quad f \text{ and } f' \in \mathcal{C}[a,b]\}$, and the norm

$$\|f\|_{1,1} = \int_a^b (|f(x)| + |f'(x)|)\, dx.$$

(a) Show there exists $M > 0$ such that for all $f \in \mathcal{C}^1[a,b]$, $\|f\|_\infty \leq M\|f\|_{1,1}$. *Hint: Use the Fundamental Theorem of Calculus.*

(b) Show that if $\{f_n\}_{n=1}^\infty$ is a sequence from $\mathcal{C}^1$ and $f_n \to g$ in $\| \circ \|_{1,1}$ then $f_n \to g$ point wise.

(c) Define $W^{1,1}$ to be the set of all sequences from C^1 which are Cauchy in the norm $\| \circ \|_{1,1}$. Show that for any sequence $\{f_n\}$ from C^1 which is Cauchy in $\| \circ \|_{1,1}$ there is a $g \in \mathcal{C}[a,b]$ such that

$$\|f_n - g\|_1 \to 0.$$

For this reason we say that

$$W^{1,1} \subset \mathcal{C}[a,b].$$

3)(a) Let $f \in L^1[a,b]$ and $g \in W^{1,1}[a,b]$. Show that the product $fg \in L^1[a,b]$. That is, if f is represented by the $\| \circ \|_1$ Cauchy sequence $\{f_n\} \subset \mathcal{C}[a,b]$ and g by the $\| \circ \|_{1,1}$ Cauchy sequence $\{g_n\} \subset \mathcal{C}^1[a,b]$, then the sequence $\{h_n\}$ where $h_n = f_n g_n$ is contained in $C[a,b]$ and is Cauchy in $\| \circ \|_1$.

(b) In part (a), show that if g is merely in $L^1[a,b]$, then the product fg may not be in $L^1[a,b]$. That is find two sequences $\{f_n\}$ and $\{g_n\}$, both from $\mathcal{C}[a,b]$ and both Cauchy in $\| \circ \|_1$ such that the "product" $\{f_n g_n\}$ is not Cauchy in $\| \circ \|_1$.