M421 HW 1

Due Friday Sept. 8

From Wade

Section	Page Number	Problems
5.1	114-115	2b (note P_n given in 2a), 3, 4, 5, 6

Non-book Exercises

1) Complete the proof of Remark 5.7. Show that if $f : [a,b] \mapsto \mathbf{R}$ is bounded, and $P, Q \in \mathcal{P}[a,b]$ satisfy $Q \supseteq P$ then $U(f,Q) \leq U(f,P)$.

2) Negate the following statements:

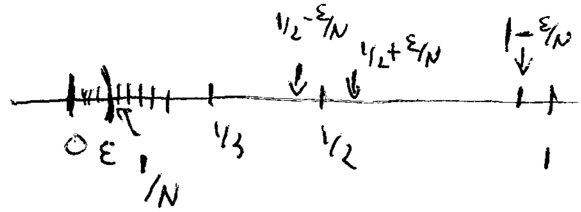
(a) It rains every wednesday.
(b) If it rains on wednesday then it will snow on thursday.
(c) For every $\epsilon > 0$ there exists a δ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$.
(d) $\sup_{x\in[a,b]} f(x) < \infty$.
(e) f is Riemann integrable, that is: $\forall \epsilon > 0$ there exists $P \in \mathcal{P}[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

5.1 #3 Let $E = \{1/n \mid n \in \mathbb{N}\}$. Prove that

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & x \in [0,1] \setminus E \end{cases}$$

is integrable on $[0,1]$. What is the value of $\int_0^1 f(x)\,dx$?

Fix $\varepsilon > 0$, construct $P \in \mathcal{P}[0,1]$ $\ni$ $\mathcal{U}(f,P) - L(f,P) < \varepsilon$.



Let N be $\ni$ $\frac{1}{N} > \varepsilon$ and $\varepsilon \geq \frac{1}{N+1}$.

Let $P = \{0, \varepsilon, 1\} \cup \left\{ \frac{1}{k} - \frac{\varepsilon}{N}, \frac{1}{k} + \frac{\varepsilon}{N} \;\middle|\; k = 1, 2, \dots, N \right\}$.

Then $L(f,P) = 0$ since $m_j^f = 0$ on all intervals (x_j, x_{j+1})

while $M_j^f = \begin{matrix} 1 \\ 0 \end{matrix}$ on $[0,\varepsilon)$, $\left(\frac{1}{2} - \frac{\varepsilon}{N}, \frac{1}{2} + \frac{\varepsilon}{N}\right)$, and $(1 - \varepsilon/N, 1)$ otherwise

there are N of these

$$\mathcal{U}(f,P) = \sum_{j=0}^{n} M_j^f (x_j - x_{j-1}) \leq \varepsilon + N \cdot 1 \cdot \frac{2\varepsilon}{N}$$

$$\leq 3\varepsilon$$

$\mathcal{U}(f,P) - L(f,P) \leq 3\varepsilon$ $\Rightarrow$ f is integrable since $\varepsilon > 0$ was arbitrary

Since f is integrable

$$\int_a^b f\,dx = \underline{\int_a^b} f\,dx = \sup_{P \in \mathcal{P}} L(f,P) = \sup 0 = 0$$

#5) Suppose that $f \in \mathscr{C}[a,b]$, show that

5.1 $$\int_a^c f(x)\,dx = 0$$

$\forall c \in [a,b]$ iff $f(x) = 0$ $\forall x \in [a,b]$

$\boxed{\Leftarrow}$ If $f(x) = 0$ $\forall x \in [a,b]$ then the result follows from the comparison principle.

$\boxed{\Rightarrow}$ We have shown that if $c \in [a,b]$ and $f \in I[a,b]$ then $f \in I[a,c] \cap I[c,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Now Assume $f(x_0) \neq 0$ for some $x_0 \in (a,b)$, then

$$\int_a^{x_0+\delta} f(x)\,dx = \int_a^{x_0-\delta} f(x)\,dx = 0$$

$$\Rightarrow \int_a^{x_0+\delta} f\,dx - \int_a^{x_0-\delta} f\,dx = \int_{x_0-\delta}^{x_0+\delta} f\,dx = 0$$

for all δ small enough that $x_0+\delta,\ x_0-\delta \in [a,b]$.

WLOG $f(x_0) > 0$. Take δ so small that

$$f(x) > \frac{f(x_0)}{2} \quad \text{on} \quad (x_0-\delta, x_0+\delta)$$

this is possible since f is continuous at x_0.

then $\forall P \in \mathscr{P}[x_0-\delta, x_0+\delta]$

$$L(f,P) = \sum_{j=0}^{n} m_j (x_{j+1}-x_j) \geq \sum_{j=0}^{n} \frac{f(x_0)}{2}(x_{j+1}-x_j)$$
$$= \frac{f(x_0)}{2}(x_0+\delta - x_0-\delta) = \delta f(x_0)$$

but $0 = \int_{x_0-\delta}^{x_0+\delta} f = \underline{\int}_{x_0-\delta}^{x_0+\delta} f \geq L(f,P) \geq \delta f(x_0) > 0$

Contradiction! $\therefore$ $f(x_0) = 0$ $\forall x_0 \in (a,b)$